Mathematics and Its Applications

Aurel Bejancu and
Hani Reda Farran

Foliations and Geometric Structures

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Preface

The theory of foliations of manifolds was created in the forties of the last century by Ch. Ehresmann and G. Reeb [ER44]. Since then, the subject has enjoyed a rapid development and thousands of papers investigating foliations have appeared. A list of papers and preprints on foliations up to 1995 can be found in Tondeur [Ton97].

Due to the great interest of topologists and geometers in this rapidly evolving theory, many books on foliations have also been published one after the other. We mention, for example, the books written by: I. Tamura [Tam76], G. Hector and U. Hirsch [HH83], B. Reinhart [Rei83], C. Camacho and A.L. Neto [CN85], H. Kitahara [Kit86], P. Molino [Mol88], Ph. Tondeur [Ton88], [Ton97], V. Rovenskii [Rov98], A. Candel and L. Conlon [CC03]. Also, the survey written by H.B. Lawson, Jr. [Law74] had a great impact on the development of the theory of foliations.

So it is natural to ask: why write yet another book on foliations? The answer is very simple. Our areas of interest and investigation are different. The main theme of this book is to investigate the interrelations between foliations of a manifold on one hand, and the many geometric structures that the manifold may admit on the other hand. Among these structures we mention: affine, Riemannian, semi–Riemannian, Finsler, symplectic, and contact structures. We also mention that, for the first time in the literature, we present in a book form results on degenerate (null, light–like) foliations of semi–Riemannian manifolds. Using these structures one obtains very interesting classes of foliations whose geometry is worth investigating. There are still many aspects of this geometry that can be promising areas for more research. We hope that the body of geometry and techniques developed in this book will show the richness of the subjects waiting to be studied further, and will present the means and tools needed for such investigations. Another point that makes our book different from the others, is that we use only two (adapted) linear connections which have been considered first by G. Vrănceanu [VG31], [VG57], and J.A. Schouten and E.R. Van Kampen [SVK30] for studying the geometry of non–holonomic spaces. Thus our study appears as a continuation of the study of non–holonomic spaces (non–integrable distributions) to foliations (integrable distributions). Furthermore, the book shows how the scientific material developed for foliations can be used in some applications to physics.

We hope that the audience of this book will include graduate students who want to be introduced to the geometry of foliations, researchers interested in foliations and geometric structures, and physicists interested in gauge theory and its generalizations.

The first chapter is devoted to the geometry of distributions. We present here a modern approach to the geometry of non–holonomic manifolds, stressing the importance of the role of the Schouten–Van Kampen connection and the Vrănceanu connection for understanding this geometry.

The theory of foliations is introduced in Chapter 2. We give the different approaches to this theory with examples showing that foliations on manifolds appear in many natural ways. A tensor calculus is then built on foliated manifolds to enable us to study the geometry of both the foliations and the ambient manifolds.

Foliations on semi–Riemannian manifolds are studied in Chapter 3. Important classes of such foliations are investigated. These include foliations with bundle–like metrics, totally geodesic, totally umbilical, minimal, symmetric and transversally symmetric foliations.

Chapter 4 deals with parallelism of foliations on semi–Riemannian manifolds. Here we study both the degenerate and non–degenerate foliations on semi–Riemannian manifolds. The situation of parallel partially–null foliations is still very far from being fully understood. We hope that our exposition stimulates further investigations trying to tackle the remaining unsolved problems.

More geometric structures on foliated manifolds are displayed in the fifth chapter. These include Lagrange foliations on symplectic manifolds, Legendre foliations on contact manifolds, foliations on the tangent bundles of Finsler manifolds, and foliations on CR–submanifolds. It is interesting to note that in Section 5.3 we develop a new method for studying the geometry of a Finsler manifold. This is mainly based on the Vrănceanu connection whose local coefficients determine all classical Finsler connections.

The last chapter is dedicated to applications. Since any vector bundle admits a natural foliation by fibers, we use the theory of foliations to develop a gauge theory on the total space of a vector bundle. We investigate the invariance of Lagrangians and obtain the equations of motion and conservation laws for the full Lagrangian. Finally, we derive the Bianchi identities for the strength fields of the gauge fields.

The preparation of the manuscript took longer than originally planned. We would like to thank both Kluwer and Springer publishers for their patience, cooperation and understanding.

We are also grateful to all the authors of books and articles whose work on foliations has been used by us in preparing the book. Many thanks go to the staff of the library "Seminarul Matematic Al. Myller" from Iaşi (Romania), for providing us with some references on non–holonomic spaces published in the first half of the last century.

It is a great pleasure for us to thank Mrs. Elena Mocanu for the excellent job of typing the manuscript. Her dedication and professionalism are very much appreciated. Finally, our thanks are due, as well, to Bassam Farran for his continuous help with the technical aspects of producing the typescript.

Kuwait
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1

GEOMETRY OF DISTRIBUTIONS ON A MANIFOLD

In the third decade of the last century, Vrănceanu [VG26a] and Horak [Hor27] introduced independently the notion of non–holonomic manifold as a need for a geometric interpretation of non–holonomic mechanical systems. We present here a modern approach to the geometry of non–holonomic manifolds as manifolds endowed with non–integrable distributions, and extend this study to almost product manifolds. Our approach is mainly based on adapted linear connections, stressing the important role of Schouten–Van Kampen and Vrănceanu connections for understanding the geometry of distributions, in general, and the geometry of non–holonomic manifolds, in particular. When a semi–Riemannian metric is considered on the manifold, we compare the intrinsic and induced connections on a semi–Riemannian manifold, and get the local structure of the manifold when these connections coincide. By using both the Schouten–Van Kampen and Vrănceanu connections we obtain the fundamental equations and some interesting evaluations for sectional curvatures of non–holonomic manifolds. In particular, we find a large class of Riemannian non–holonomic manifolds of Vrănceanu positive constant curvature. Finally, we present a method to study the geometry of degenerate distributions of codimension one on a proper semi–Riemannian manifold.

Our approach to the geometry of distributions on a manifold via Schouten–Van Kampen and Vrănceanu connections is given, not only because of its importance for its own right, but also because of the crucial role it will play throughout the book in studying foliations on manifolds.

1.1 Distributions on a Manifold

Let M be an $(n+p)-$dimensional paracompact smooth manifold and TM be the tangent bundle of M. Denote by π the canonical projection of TM on M and by T_xM the fiber at $x \in M$, i.e., $T_xM = \pi^{-1}(x)$. A coordinate system (local chart) in M is denoted by $\{(\mathcal{U}, \varphi) : (x^1, ..., x^{n+p})\}$ or briefly $\{(\mathcal{U}, \varphi) : (x^a)\}$, where $\mathcal{U}$ is an open subset of M, $\varphi : \mathcal{U} \longrightarrow \mathbb{R}^{n+p}$ is a diffeomorphism of $\mathcal{U}$ onto $\varphi(\mathcal{U})$, and $(x^1, ..., x^{n+p}) = \varphi(x)$ for any $x \in \mathcal{U}$. For any point $x \in \mathcal{U}$, we say that the coordinate system $\{(\mathcal{U}, \varphi) : (x^a)\}$ is about x. The coordinate system $\{(\mathcal{U}, \varphi) : (x^a)\}$ in M defines a coordinate system $\{(\mathcal{U}^*, \Phi) : (x^1, ..., x^{n+p}, v^1, ..., v^{n+p})\}$ in TM, where $\mathcal{U}^* = \pi^{-1}(\mathcal{U})$ and $\Phi : \mathcal{U}^* \longrightarrow \mathbb{R}^{2(n+p)}$ is a diffeomorphism of $\mathcal{U}^*$ onto $\varphi(\mathcal{U}) \times \mathbb{R}^{n+p}$ and $(x^1, ..., x^{n+p}, v^1, ..., v^{n+p}) = \Phi(v_x)$ for any $x \in \mathcal{U}$ and $v_x \in T_xM$.

Next, we consider a vector subbundle $\mathcal{D}$ of TM of rank n. Thus for each $x \in M$ there exists a local chart $(\mathcal{U}, \varphi)$ on M at x such that the corresponding local chart $(\mathcal{U}^*, \Phi)$ on TM satisfies the condition $\Phi(\mathcal{U}^* \cap \mathcal{D}) = \varphi(\mathcal{U}) \times \mathbb{R}^n$. Then each fiber $\mathcal{D}_x$ over $x \in M$ is an $n-$dimensional subspace of T_xM, and the total space of the vector bundle $\pi : \mathcal{D} \longrightarrow M$ becomes a $(2n+p)-$dimensional submanifold of TM. We say that $\mathcal{D}$ is an $n-$**distribution** ($n-$**plane field** or $n-$**differential system**) on M.

A slightly different approach to distributions may be achieved by starting with the **Grassmann bundle** $G_n(M)$ over M. For any $x \in M$ the Grassmann manifold $G_n(x)$ consists of all $n-$dimensional vector subspaces of the tangent space T_xM. Then

$$G_n(M) = \bigcup_{x \in M} G_n(x),$$

is an $(n + p + np)-$dimensional manifold, since each fiber $G_n(x)$ is an $np-$dimensional manifold. Clearly, any smooth section of $G_n(M)$ is an $n-$distribution and conversely, any $n-$distribution defines a section of $G_n(M)$.

We do not explore here the difficult problem of the existence of distributions on a manifold. We only mention that if M is a compact manifold and its Euler number $\mathcal{X}(M)$ is zero, then there exists on M a $1-$distribution. Thus any odd$-$dimensional sphere S^m with $m \geq 3$ admits a $1-$distribution. Also, we note that the only compact surfaces with $1-$distributions are the torus and the Klein bottle. Since fibers of a $1-$distribution are lines, we refer to a $1-$distribution as a **line field**.

As we have seen, a distribution on M is globally given either as a vector subbundle of TM or as a global section of $G_n(M)$. However, most of the problems encountering distributions have a local character. Here we present two ways to define a distribution on M by some geometric objects that are locally defined on M.

First, suppose that on each coordinate neighbourhood $\mathcal{U}$ in M there exist n linearly independent smooth vector fields $\{E_1, ..., E_n\}$. Then the mapping

$$x \longrightarrow \mathcal{D}_x = \text{span}\{E_{1x}, ..., E_{nx}\}, \ x \in \mathcal{U},$$

defines an $n-$distribution on $\mathcal{U}$. Now, we assume that for any two coordinate neighbourhoods $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ with $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$, the vector fields $\{\widetilde{E}_1, ..., \widetilde{E}_n\}$ and $\{E_1, ..., E_n\}$ are related by

$$\widetilde{E}_i = a_i^j E_j, \tag{1.1}$$

where a_i^j are smooth functions on $\mathcal{U} \cap \widetilde{\mathcal{U}}$ such that $[a_i^j(x)]$ is a non–singular $n \times n$ matrix for any $x \in \mathcal{U} \cap \widetilde{\mathcal{U}}$. In this way the two distributions on $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ agree on $\mathcal{U} \cap \widetilde{\mathcal{U}}$ and therefore we have a distribution on M. Conversely, it is easy to see that any $n-$distribution on M is locally represented by n linearly independent smooth vector fields satisfying (1.1) on the intersection of two coordinate neighbourhoods. Though we do not write the adjective "smooth" to the noun "distribution" we always understand that all local representative vector fields of a distribution are smooth vector fields.

A distribution on a manifold can also be locally defined using a differential system. This is done as follows. We assume that on each coordinate neighbourhood $\mathcal{U} \subset M$ there exist p linearly independent smooth 1–forms $\{\omega^\alpha\}$, $\alpha \in \{n+1, ..., n+p\}$. Then for any $x \in \mathcal{U}$ we consider $\mathcal{D}_x$ as the $n-$dimensional subspace of T_xM consisting of solutions X_x of the system

$$\omega^{n+1}(X) = 0, ..., \omega^{n+p}(X) = 0. \tag{1.2}$$

Next we add the condition that the 1–forms $\{\omega^{n+1}, ..., \omega^{n+p}\}$ and $\{\widetilde{\omega}^{n+1}, ..., \widetilde{\omega}^{n+p}\}$ on $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ satisfy

$$\widetilde{\omega}^\alpha = \Lambda_\beta^\alpha \omega^\beta, \quad \text{on } \mathcal{U} \cap \widetilde{\mathcal{U}}, \tag{1.3}$$

where Λ_β^α are smooth functions on $\mathcal{U} \cap \widetilde{\mathcal{U}}$ such that $[\Lambda_\beta^\alpha(x)]$ is a non–singular $p \times p$ matrix for any $x \in \mathcal{U} \cap \widetilde{\mathcal{U}}$. Then the mapping $\mathcal{D} : x \rightarrow \mathcal{D}_x \in G_n(x)$ defines an $n-$distribution on M. The converse is also true, that is, any $n-$distribution on M is given locally by a differential system (1.2) whose representative 1–forms are related by (1.3).

If not stated otherwise, we shall use throughout this chapter the following ranges for indices: $i, j, k, ... \in \{1, ..., n\}$; $\alpha, \beta, \gamma, ... \in \{n + 1, ..., n + p\}$; $a, b, c, ... \in \{1, ..., n + p\}$.

The integrability problem for distributions is very important. A complete study of this problem is going to be presented in the next chapter (see Section 2.1). Here we only give some definitions and discuss their equivalence.

Let $\mathcal{D}$ be an $n-$distribution on M. Then a $k-$dimensional submanifold N of M, $0 < k \leq n$, is said to be an **integral manifold** of $\mathcal{D}$, if $T_xN \subset \mathcal{D}_x$ for any $x \in N$. Thus the maximum dimension of N is n. Now, we say that $\mathcal{D}$ is an **integrable distribution** if for any point $x \in M$ there exists a local chart $\{(\mathcal{U}, \varphi) : (x^1, ..., x^n, x^{n+1}, ..., x^{n+p})\}$ on M such that all the submanifolds of $\mathcal{U}$ given by the equations

$$x^{n+1} = \text{constant}, ..., x^{n+p} = \text{constant}, \tag{1.4}$$

are integral manifolds of $\mathcal{D}$.

A connected submanifold given by (1.4) is called a **local leaf** (**plaque**) of $\mathcal{D}$ (details can be seen in Section 2.1). In this case any connected integral manifold of $\mathcal{D}$ lying in $\mathcal{U}$ is a submanifold of one of the local leaves of $\mathcal{D}$. Based on the above definition we can state the following.

Theorem 1.1. *Let $\mathcal{D}$ be an n–distribution on M. Then the following assertions are equivalent:*

(i) *$\mathcal{D}$ is an integrable distribution.*
(ii) *For any $x \in M$ there exists a local chart $\{(\mathcal{U}, \varphi) : (x^a)\}$ such that $\mathcal{D}$ is given on $\mathcal{U}$ by the differential system*

$$dx^{n+1} = 0, ..., dx^{n+p} = 0. \tag{1.5}$$

(iii) *For any $x \in M$ there exists a local chart $\{(\mathcal{U}, \varphi) : (x^a)\}$ such that*

$$\mathcal{D} = \text{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}, \quad \textit{on } \mathcal{U}. \tag{1.6}$$

Next, let X and Y be two vector fields on M. Then their **Lie bracket** $[X, Y]$ is a vector field defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall f \in F(M). \tag{1.7}$$

Locally, the Lie bracket is written as follows

$$[X, Y] = \left(X^a \frac{\partial Y^b}{\partial x^a} - Y^a \frac{\partial X^b}{\partial x^a}\right) \frac{\partial}{\partial x^b}, \tag{1.8}$$

where $X = X^a \dfrac{\partial}{\partial x^a}$ and $Y = Y^a \dfrac{\partial}{\partial x^a}$. Now, we say that a vector field X on M **lies** in $\mathcal{D}$ if $X(x) \in \mathcal{D}_x$, for all $x \in M$. If $\Gamma(\mathcal{D})$ denotes the $F(M)$–module of smooth sections of $\mathcal{D}$, then we use the notation $X \in \Gamma(\mathcal{D})$ to indicate that X lies in $\mathcal{D}$. We say that $\mathcal{D}$ is an **involutive distribution** if $[X, Y] \in \Gamma(\mathcal{D})$ for any $X, Y \in \Gamma(\mathcal{D})$. At this point we only mention that $\mathcal{D}$ is integrable if and only if it is involutive. This is the famous theorem of Frobenius which will be proved in Section 2.1.

In the present chapter we will be concerned with the geometry of distributions in general, that is, they do not need to be integrable. A pair $(M, \mathcal{D})$, where M is a manifold and $\mathcal{D}$ is a non–integrable distribution on M, is called a **non–holonomic manifold**. The concept of "non–holonomic space" in a Riemannian manifold has been introduced in 1926 by Vrănceanu [VG26a], [VG26b] and independently by Horak [Hor27] in 1927 as a need for a geometric interpretation of non–holonomic mechanical systems. In 1928 Schouten [Sch28] considered non–holonomic spaces in a manifold with a linear connection. A great deal of research has been devoted to the study of the geometry of non–holonomic spaces in Riemannian manifolds, and in manifolds with linear connections, in general. Several references published in the first half of the 20th century can be found in Schouten [Sch54].

The purpose of this chapter is to revisit this rather forgotten area of differential geometry. In addition to the classical coordinate–base approach, we will exploit modern coordinate–free techniques. The information we present here will be used later in the book in our search for results that shed more light on the geometry of foliated manifolds. In this respect, it is worth mentioning that the linear connections introduced by Vrănceanu [VG31] and Schouten and Van Kampen [SVK30] on non–holonomic manifolds will be considered on almost product manifolds, and thus they will have an important role in studying foliations on Riemannian (semi–Riemannian) manifolds.

If a distribution $\mathcal{D}$ on M is given, then a complementary distribution $\mathcal{D}'$ to $\mathcal{D}$ in TM can be obtained. Indeed, since M is paracompact and of differentiability class C^∞, there exists on M a Riemannian metric of class C^∞. Then we can take $\mathcal{D}'$ as the complementary orthogonal distribution to $\mathcal{D}$ with respect to that metric. Thus we are entitled to consider, in the first stage of our study, a pair of complementary distributions $(\mathcal{D}, \mathcal{D}')$ on M, that is, TM has the decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}'. \tag{1.9}$$

Later on (see Sections 1.5, 1.6 and 1.7) we will see the contribution of a Riemannian (semi–Riemannian) metric on M to the study of the geometry of the pair $(\mathcal{D}, \mathcal{D}')$.

Based on the above discussion we consider on M two complementary distributions $\mathcal{D}$ and $\mathcal{D}'$. Denote by Q and Q' the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then we have

$$\begin{aligned} &\text{(a) } Q^2 = Q, && \text{(b) } Q'^2 = Q', \\ &\text{(c) } QQ' = Q'Q = 0, && \text{(d) } Q + Q' = I, \end{aligned} \tag{1.10}$$

where I is the identity morphism on TM. Now we define the tensor field F of type $(1,1)$ by

$$F = Q - Q'. \tag{1.11}$$

It follows that F is an **almost product structure** on M, that is, F satisfies

$$F^2 = I. \tag{1.12}$$

For this reason we call $(M, \mathcal{D}, \mathcal{D}')$ an **almost product manifold**. Next, from (1.10d) and (1.11) we deduce that

$$\text{(a) } Q = \frac{1}{2}(I + F) \quad \text{and} \quad \text{(b) } Q' = \frac{1}{2}(I - F). \tag{1.13}$$

Now, we note that at any point $x \in M$, $\mathcal{D}_x$ and $\mathcal{D}'_x$ coincide with the eigenspaces in T_xM corresponding to the eigenvalues $+1$ and -1 of F_x, respectively. Indeed, if $X_x \in T_xM$ and $F_x(X_x) = X_x$, then from (1.13a) we deduce that $Q_x(X_x) = X_x$, that is, $X_x \in \mathcal{D}_x$. Conversely, if $X_x \in \mathcal{D}_x$ then there exists $Y_x \in T_xM$ such that $Q_x(Y_x) = X_x$. Then, by using (1.11), (1.10a) and (1.10c) we obtain $F_x(X_x) = X_x$. The corresponding property for $\mathcal{D}'_x$ is obtained similarly. As a conclusion we write

$$\begin{aligned} &(a)\ \Gamma(\mathcal{D}) = \{X \in \Gamma(TM) : FX = X\}, \\ &(b)\ \Gamma(\mathcal{D}') = \{X \in \Gamma(TM) : FX = -X\}. \end{aligned} \tag{1.14}$$

Next, we suppose that $\mathcal{D}$ and $\mathcal{D}'$ are locally represented on a coordinate neighbourhood $\mathcal{U} \subset M$ by vector fields $\{E_i\}$ and $\{E_\alpha\}$ respectively. Then we call $\{E_A\} = \{E_i, E_\alpha\}$, $A \in \{1, ..., n+p\}$, a **non–holonomic frame field** on $\mathcal{U}$. Thus from now on, in this chapter, the indices $A, B, C, ...$ have the same range $\{1, ..., n+p\}$ as the indices $a, b, c, ...$, but the latters are used as indices for local components of geometric objects defined by means of the holonomic frame and coframe fields $\left\{\dfrac{\partial}{\partial x^a}\right\}$ and $\{dx^a\}$ on $\mathcal{U}$. According to the definition of a distribution on a manifold, the transformation of non–holonomic frame fields on $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$ is given by

$$(a)\ \widetilde{E}_i = a_i^j E_j, \qquad (b)\ \widetilde{E}_\alpha = a_\alpha^\beta E_\beta, \tag{1.15}$$

where $[a_i^j]$ and $[a_\alpha^\beta]$ are $n\times n$ and $p\times p$ non–singular matrices respectively. Now, we consider the natural field of frames $\left\{\dfrac{\partial}{\partial x^a}\right\}$ on M and put

$$(a)\ E_A = E_A^a \frac{\partial}{\partial x^a} \quad \text{and} \quad (b)\ \frac{\partial}{\partial x^a} = \bar{E}_a^A E_A. \tag{1.16}$$

Then taking into account that the $(n+p)\times(n+p)$ matrices $[E_A^a]$ and $[\bar{E}_a^A]$ are inverses for each other we deduce that

$$\begin{aligned} &(a)\ \bar{E}_a^i E_i^b + \bar{E}_a^\alpha E_\alpha^b = \delta_a^b, \\ &(b)\ E_\alpha^a \bar{E}_a^\beta = \delta_\alpha^\beta, \\ &(c)\ E_i^a \bar{E}_a^j = \delta_i^j, \\ &(d)\ E_i^a \bar{E}_a^\alpha = 0, \\ &(e)\ E_\alpha^a \bar{E}_a^i = 0. \end{aligned} \tag{1.17}$$

The dual frame field $\{\omega^A\} = \{\omega^i, \omega^\alpha\}$ to the non–holonomic frame field $E_A = \{E_i, E_\alpha\}$ is called the **dual non–holonomic coframe field** to $\{E_A\}$. Then the distributions $\mathcal{D}$ and $\mathcal{D}'$ are locally defined by the differential systems

$$\omega^\alpha = 0, \quad \alpha \in \{n+1, ..., n+p\}, \tag{1.18}$$

and

$$\omega^i = 0, \quad i \in \{1, ..., n\}, \tag{1.19}$$

respectively.

1.2 Adapted Linear Connections on Almost Product Manifolds

Let $\mathcal{D}$ be an $n-$distribution on an $(n+p)-$dimensional manifold M. A linear connection ∇^* on M is said to be **adapted** to $\mathcal{D}$ if

$$\nabla^*_X U \in \Gamma(\mathcal{D}), \quad \forall\, X \in \Gamma(TM),\ U \in \Gamma(\mathcal{D}).$$

Now, if $\mathcal{D}'$ is a $p-$distribution on M complementary to $\mathcal{D}$, then $(M, \mathcal{D}, \mathcal{D}')$ is an almost product manifold as we have seen in Section 1.1. We call $\mathcal{D}$ the **structural distribution** and $\mathcal{D}'$ a **transversal distribution**. These names were introduced by Vaisman [Vai71] when $\mathcal{D}$ is a distribution on a Riemannian manifold and $\mathcal{D}'$ is its orthogonal complement.

A linear connection ∇^* on an almost product manifold $(M, \mathcal{D}, \mathcal{D}')$ is said to be an **adapted linear connection** if it is adapted to both distributions $\mathcal{D}$ and $\mathcal{D}'$. Thus ∇^* is adapted if and only if the following conditions are satisfied:

$$\nabla^*_X QY \in \Gamma(\mathcal{D}), \quad \forall\, X, Y \in \Gamma(TM), \tag{2.1}$$

and

$$\nabla^*_X Q'Y \in \Gamma(\mathcal{D}'), \quad \forall\, X, Y \in \Gamma(TM), \tag{2.2}$$

where Q and Q' stand, as in the first section, for projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}'$ respectively. It is easy to see that an adapted linear connection ∇^* defines two linear connections ∇ and ∇' on $\mathcal{D}$ and $\mathcal{D}'$ respectively, by

$$\begin{aligned} &\text{(a) } \nabla_X QY = \nabla^*_X QY, \quad \text{and} \\ &\text{(b) } \nabla'_X Q'Y = \nabla^*_X Q'Y, \forall\, X, Y \in \Gamma(TM). \end{aligned} \tag{2.3}$$

Conversely, if ∇ and ∇' are two linear connections on $\mathcal{D}$ and $\mathcal{D}'$ respectively, then we construct an adapted linear connection ∇^* on $(M, \mathcal{D}, \mathcal{D}')$, by the formula

$$\nabla^*_X Y = \nabla_X QY + \nabla'_X Q'Y, \quad \forall\, X, Y \in \Gamma(TM). \tag{2.4}$$

Moreover, the restrictions of ∇^*_X to $\Gamma(\mathcal{D})$ and $\Gamma(\mathcal{D}')$ are exactly ∇_X and ∇'_X respectively. Thus, by the above discussion we state the following.

Theorem 2.1. *There exists on $(M, \mathcal{D}, \mathcal{D}')$ an adapted linear connection ∇^* if and only if there exists a pair (∇, ∇'), where ∇ and ∇' are linear connections on $\mathcal{D}$ and $\mathcal{D}'$ respectively.*

An adapted linear connection on $(M, \mathcal{D}, \mathcal{D}')$ can also be characterized by means of the almost product structure F given by (1.11) and as well by the projection morphisms Q and Q'. To state this we give the following definition. We say that F is **parallel** with respect to a linear connection $\widetilde{\nabla}$ on M if its covariant derivative with respect to $\widetilde{\nabla}$ vanishes, i.e., we have

$$(\widetilde{\nabla}_X F)Y = \widetilde{\nabla}_X FY - F(\widetilde{\nabla}_X Y) = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{2.5}$$

The same definition applies for Q and Q'. Then the following theorem can be easily proved.

Theorem 2.2. *Let ∇^* be a linear connection on the almost product manifold $(M, \mathcal{D}, \mathcal{D}')$. Then the following assertions are equivalent:*

(i) *∇^* is an adapted linear connection.*
(ii) *The almost product structure F is parallel with respect to ∇^*.*
(iii) *The projection morphisms Q and Q' are parallel with respect to ∇^*.*

Next, we would like to present some local characterizations of the linear connections on $\mathcal{D}$ and $\mathcal{D}'$, and therefore of the adapted linear connections on $(M, \mathcal{D}, \mathcal{D}')$. To this end, we consider the non–holonomic frame field $\{E_A\} = \{E_i, E_\alpha\}$ on $\mathcal{U} \subset M$. Then for any smooth function f on M we define

$$\begin{aligned} &\text{(a)}\ f_{|\alpha} = E_\alpha(f) = E_\alpha^a \frac{\partial f}{\partial x^a}, \quad \text{and} \\ &\text{(b)}\ f_{\|i} = E_i(f) = E_i^a \frac{\partial f}{\partial x^a}. \end{aligned} \tag{2.6}$$

We call $f_{|\alpha}$ and $f_{\|i}$ the **transversal non–holonomic derivative** and **structural non–holonomic derivative** of f with respect to the non–holonomic frame field $\{E_A\}$. Now, let ∇ and ∇' be linear connections on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then, locally on $\mathcal{U} \subset M$ we put

$$\text{(a)}\ \nabla_{E_j} E_i = \Gamma_i{}^k{}_j E_k, \qquad \text{(b)}\ \nabla_{E_\alpha} E_i = \Gamma_i{}^k{}_\alpha E_k, \tag{2.7}$$

and

$$\text{(a)}\ \nabla'_{E_j} E_\alpha = \Gamma'_\alpha{}^\beta{}_j E_\beta, \quad \text{(b)}\ \nabla'_{E_\gamma} E_\alpha = \Gamma'_\alpha{}^\beta{}_\gamma E_\beta. \tag{2.8}$$

We perform a transformation of non–holonomic frame fields, and by using (1.15), (2.7) and (2.8) we obtain

$$\begin{aligned} &\text{(a)}\ \widetilde{\Gamma}_s{}^h{}_t a_h^k = (\Gamma_i{}^k{}_j a_s^i + (a_s^k)_{\|j}) a_t^j, \\ &\text{(b)}\ \widetilde{\Gamma}_s{}^h{}_\gamma a_h^k = (\Gamma_i{}^k{}_\alpha a_s^i + (a_s^k)_{|\alpha}) a_\gamma^\alpha, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\text{(a)}\ \widetilde{\Gamma}'_\nu{}^\varepsilon{}_j a_\varepsilon^\beta = (\Gamma'_\alpha{}^\beta{}_i a_\nu^\alpha + (a_\nu^\beta)_{\|i}) a_j^i, \\ &\text{(b)}\ \widetilde{\Gamma}'_\nu{}^\varepsilon{}_\mu a_\varepsilon^\beta = (\Gamma'_\alpha{}^\beta{}_\gamma a_\nu^\alpha + (a_\nu^\beta)_{|\gamma}) a_\mu^\gamma. \end{aligned} \tag{2.10}$$

Conversely, if on each $\mathcal{U} \subset M$ there exist functions $(\Gamma_i{}^k{}_j, \Gamma_i{}^k{}_\alpha)$ and $(\Gamma'_\alpha{}^\beta{}_i, \Gamma'_\alpha{}^\beta{}_\gamma)$ satisfying (2.9) and (2.10) with respect to the transformation (1.15) of non–holonomic frame fields, then the differential operators ∇ and ∇' given by (2.7) and (2.8) define linear connections on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Thus we may state the following.

Theorem 2.3.

(i) *There exists a linear connection on* $\mathcal{D}$ *if and only if on each coordinate neighbourhood* $\mathcal{U} \subset M$ *there exist* $n^2(n+p)$ *functions* $({\Gamma_i}^k{}_j, {\Gamma_i}^k{}_\alpha)$ *satisfying* (2.9) *with respect to* (1.15).

(ii) *There exists a linear connection on* $\mathcal{D}'$ *if and only if on each coordinate neighbourhood* $\mathcal{U} \subset M$ *there exist* $p^2(n+p)$ *functions* $({\Gamma'_\alpha}{}^\beta{}_i, {\Gamma'_\alpha}{}^\beta{}_\gamma)$ *satisfying* (2.10) *with respect to* (1.15).

The next corollary follows from Theorems 2.1 and 2.3.

Corollary 2.4. *The exists an adapted linear connection on* M *if and only if on each* $\mathcal{U} \subset M$ *there exist* $(n^2+p^2)(n+p)$ *functions* $({\Gamma_i}^k{}_j, {\Gamma_i}^k{}_\alpha, {\Gamma'_\alpha}{}^\beta{}_i, {\Gamma'_\alpha}{}^\beta{}_\gamma)$ *satisfying* (2.9) *and* (2.10) *with respect to* (1.15).

Thus an adapted linear connection ∇^* on M is locally given by

$$\begin{aligned} &\text{(a) } \nabla^*_{E_j} E_i = {\Gamma_i}^k{}_j E_k, \quad &&\text{(b) } \nabla^*_{E_\alpha} E_i = {\Gamma_i}^k{}_\alpha E_k, \\ &\text{(c) } \nabla^*_{E_j} E_\alpha = {\Gamma'_\alpha}{}^\beta{}_j E_\beta, \quad &&\text{(d) } \nabla^*_{E_\gamma} E_\alpha = {\Gamma'_\alpha}{}^\beta{}_\gamma E_\beta, \end{aligned} \tag{2.11}$$

where the non–holonomic coefficients satisfy the conditions from Corollary 2.4.

Next, we consider an adapted linear connection $\nabla^* = ({\Gamma_i}^k{}_A, {\Gamma'_\alpha}{}^\beta{}_A)$ on $(M, \mathcal{D}, \mathcal{D}')$ and look for the non–holonomic local components of its torsion and curvature tensor fields with respect to a non–holonomic frame field. To achieve this we put:

$$\begin{aligned} &\text{(a) } Q[E_j, E_i] = {V_i}^k{}_j E_k, \quad \text{(b) } Q[E_\beta, E_\alpha] = {V_\alpha}^k{}_\beta E_k, \\ &\text{(c) } Q[E_\alpha, E_i] = -Q[E_i, E_\alpha] = {V_i}^k{}_\alpha E_k = -{V_\alpha}^k{}_i E_k, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\text{(a) } Q'[E_j, E_i] = {V'_i}{}^\beta{}_j E_\beta, \quad \text{(b) } Q'[E_\gamma, E_\alpha] = {V'_\alpha}{}^\beta{}_\gamma E_\beta, \\ &\text{(c) } Q'[E_i, E_\alpha] = -Q'[E_\alpha, E_i] = {V'_\alpha}{}^\beta{}_i E_\beta = -{V'_i}{}^\beta{}_\alpha E_\beta. \end{aligned} \tag{2.13}$$

Then we recall that the torsion tensor field T^* of the linear connection ∇^* is given by (cf. Kobayashi–Nomizu [KN63], p. 133)

$$T^*(X, Y) = \nabla^*_X Y - \nabla^*_Y X - [X, Y], \quad \forall\, X, Y \in \Gamma(TM). \tag{2.14}$$

By using the decomposition (1.9) and the non–holonomic frame field $\{E_A\}$ we set:

$$\begin{aligned} &\text{(a) } T^*(E_j, E_i) &&= {T_i}^k{}_j E_k + {T'_i}{}^\alpha{}_j E_\alpha, \\ &\text{(b) } T^*(E_\alpha, E_i) &&= -T^*(E_i, E_\alpha) = {T_i}^k{}_\alpha E_k + {T'_i}{}^\beta{}_\alpha E_\beta \\ & &&= -{T_\alpha}^k{}_i E_k - {T'_\alpha}{}^\beta{}_i E_\beta, \\ &\text{(c) } T^*(E_\gamma, E_\alpha) &&= {T_\alpha}^k{}_\gamma E_k + {T'_\alpha}{}^\beta{}_\gamma E_\beta. \end{aligned} \tag{2.15}$$

Then by direct calculations using (2.11)–(2.15) we obtain all non–holonomic components of T^* as in the next theorem.

Theorem 2.5. *Let* $\nabla^* = (\Gamma_i{}^k{}_A, \Gamma'_\alpha{}^\beta{}_A)$ *be an adapted linear connection on the almost product manifold* $(M, \mathcal{D}, \mathcal{D}')$. *Then the local components of its torsion tensor field with respect to a non–holonomic frame field* $\{E_A\}$ *are given by*

$$\begin{aligned}
&(a)\ T_i{}^k{}_j = \Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i - V_i{}^k{}_j,\\
&(b)\ T'_i{}^\alpha{}_j = -V'_i{}^\alpha{}_j,\\
&(c)\ T_i{}^k{}_\alpha = -T_\alpha{}^k{}_i = \Gamma_i{}^k{}_\alpha - V_i{}^k{}_\alpha,\\
&(d)\ T'_\alpha{}^\beta{}_i = -T'_i{}^\beta{}_\alpha = \Gamma'_\alpha{}^\beta{}_i - V'_\alpha{}^\beta{}_i,\\
&(e)\ T_\alpha{}^k{}_\beta = -V_\alpha{}^k{}_\beta,\\
&(f)\ T'_\alpha{}^\beta{}_\gamma = \Gamma'_\alpha{}^\beta{}_\gamma - \Gamma'_\gamma{}^\beta{}_\alpha - V'_\alpha{}^\beta{}_\gamma.
\end{aligned} \tag{2.16}$$

We now look for the non–holonomic local components of the curvature tensor field R^* of ∇^*, given by (cf. Kobayashi–Nomizu [KN63], p. 133)

$$R^*(X, Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z, \tag{2.17}$$

for any $X, Y, Z \in \Gamma(TM)$. To this end we first note that the $F(M)$–linear opertor $R^*(X, Y)$ on $\Gamma(TM)$ induces $F(M)$–linear operators on both $\Gamma(\mathcal{D})$ and $\Gamma(\mathcal{D}')$. This enables us to set:

$$\begin{aligned}
&(a)\ R^*(E_k, E_j)E_i = R_i{}^h{}_{jk} E_h,\\
&(b)\ R^*(E_k, E_\alpha)E_i = -R^*(E_\alpha, E_k)E_i = R_i{}^h{}_{\alpha k} E_h = -R_i{}^h{}_{k\alpha} E_h,\\
&(c)\ R^*(E_\beta, E_\alpha)E_i = R_i{}^h{}_{\alpha\beta} E_h,
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&(a)\ R^*(E_k, E_j)E_\alpha = R'_\alpha{}^\beta{}_{jk} E_\beta,\\
&(b)\ R^*(E_k, E_\gamma)E_\alpha = -R^*(E_\gamma, E_k)E_\alpha = R'_\alpha{}^\beta{}_{\gamma k} E_\beta = -R'_\alpha{}^\beta{}_{k\gamma} E_\beta,\\
&(c)\ R^*(E_\mu, E_\gamma)E_\alpha = R'_\alpha{}^\beta{}_{\gamma\mu} E_\beta.
\end{aligned} \tag{2.19}$$

The proof of the next theorem follows by direct calculations using (2.11)–(2.13) and (2.17)–(2.19).

Theorem 2.6. *Let* $\nabla^* = (\Gamma_i{}^k{}_A, \Gamma'_\alpha{}^\beta{}_A)$ *be an adapted linear connection on the almost product manifold* $(M, \mathcal{D}, \mathcal{D}')$. *Then the local components of its curvature tensor field with respect to a non–holonomic frame field* $\{E_A\}$ *are given by*

$$\begin{aligned}
&(a)\ R_i{}^h{}_{jk} = \Gamma_i{}^h{}_{j\|k} - \Gamma_i{}^h{}_{k\|j} + \Gamma_i{}^s{}_j\Gamma_s{}^h{}_k - \Gamma_i{}^s{}_k\Gamma_s{}^h{}_j \\
&\qquad - \Gamma_i{}^h{}_s V_j{}^s{}_k - \Gamma_i{}^h{}_\alpha V'_j{}^\alpha{}_k, \\
&(b)\ R_i{}^h{}_{\alpha k} = \Gamma_i{}^h{}_{\alpha\|k} - \Gamma_i{}^h{}_{k|\alpha} + \Gamma_i{}^s{}_\alpha\Gamma_s{}^h{}_k - \Gamma_i{}^s{}_k\Gamma_s{}^h{}_\alpha \\
&\qquad - \Gamma_i{}^h{}_s V_\alpha{}^s{}_k - \Gamma_i{}^h{}_\varepsilon V'_\alpha{}^\varepsilon{}_k, \\
&(c)\ R_i{}^h{}_{\alpha\beta} = \Gamma_i{}^h{}_{\alpha|\beta} - \Gamma_i{}^h{}_{\beta|\alpha} + \Gamma_i{}^s{}_\alpha\Gamma_s{}^h{}_\beta - \Gamma_i{}^s{}_\beta\Gamma_s{}^h{}_\alpha \\
&\qquad - \Gamma_i{}^h{}_s V_\alpha{}^s{}_\beta - \Gamma_i{}^h{}_\varepsilon V'_\alpha{}^\varepsilon{}_\beta,
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
&(a)\ R'_\alpha{}^\beta{}_{jk} = \Gamma'_\alpha{}^\beta{}_{j\|k} - \Gamma'_\alpha{}^\beta{}_{k\|j} + \Gamma'_\alpha{}^\varepsilon{}_j\Gamma'_\varepsilon{}^\beta{}_k - \Gamma'_\alpha{}^\varepsilon{}_k\Gamma'_\varepsilon{}^\beta{}_j \\
&\qquad - \Gamma'_\alpha{}^\beta{}_s V_j{}^s{}_k - \Gamma'_\alpha{}^\beta{}_\varepsilon V'_j{}^\varepsilon{}_k, \\
&(b)\ R'_\alpha{}^\beta{}_{\gamma k} = \Gamma'_\alpha{}^\beta{}_{\gamma\|k} - \Gamma'_\alpha{}^\beta{}_{k|\gamma} + \Gamma'_\alpha{}^\varepsilon{}_\gamma\Gamma'_\varepsilon{}^\beta{}_k - \Gamma'_\alpha{}^\varepsilon{}_k\Gamma'_\varepsilon{}^\beta{}_\gamma \\
&\qquad - \Gamma'_\alpha{}^\beta{}_s V_\gamma{}^s{}_k - \Gamma'_\alpha{}^\beta{}_\varepsilon V'_\gamma{}^\varepsilon{}_k, \\
&(c)\ R'_\alpha{}^\beta{}_{\gamma\mu} = \Gamma'_\alpha{}^\beta{}_{\gamma|\mu} - \Gamma'_\alpha{}^\beta{}_{\mu|\gamma} + \Gamma'_\alpha{}^\varepsilon{}_\gamma\Gamma'_\varepsilon{}^\beta{}_\mu - \Gamma'_\alpha{}^\varepsilon{}_\mu\Gamma'_\varepsilon{}^\beta{}_\gamma \\
&\qquad - \Gamma'_\alpha{}^\beta{}_s V_\gamma{}^s{}_\mu - \Gamma'_\alpha{}^\beta{}_\varepsilon V'_\gamma{}^\varepsilon{}_\mu.
\end{aligned} \tag{2.21}$$

Taking into account that ∇^* induces a linear connection $\nabla = (\Gamma_i{}^k{}_A)$ on $\mathcal{D}$ and a linear connection $\nabla' = (\Gamma'_\alpha{}^\beta{}_A)$ on $\mathcal{D}'$, by Theorem 2.6 we may state the following.

Corollary 2.7. *The local components of the curvature tensor fields of* ∇ *and* ∇' *with respect to a non–holonomic frame field* $\{E_A\}$ *are given by* (2.20) *and* (2.21) *respectively.*

As it is well known, a torsion tensor field is not defined, in general, for a linear connection on a vector bundle. However, by using the notion of general connection introduced by Otsuki [Ots61] we will define here a torsion tensor field for a linear connection on a distribution. To achieve this we consider a vector bundle E over M and a vector bundle morphism $P : E \longrightarrow E$. Then according to Abe [Abe85] an **Otsuki connection** (**general connection**) on E with respect to the vector bundle morphism P is a mapping $\widetilde{\nabla}$ that assigns to any $X \in \Gamma(TM)$ the differential operator

$$\widetilde{\nabla}_X : \Gamma(E) \longrightarrow \Gamma(E); \quad S \longrightarrow \widetilde{\nabla}_X S, \quad \forall\, S \in \Gamma(E),$$

satisfying the following conditions:

$$\widetilde{\nabla}_{fX+Y}(S) = f\widetilde{\nabla}_X S + \widetilde{\nabla}_Y S,$$

and

$$\widetilde{\nabla}_X(fS + S') = X(f)P(S) + f\widetilde{\nabla}_X S + \widetilde{\nabla}_X S',$$

for any $f \in F(M)$, $X, Y \in \Gamma(TM)$ and $S, S' \in \Gamma(E)$. It is easy to see that $\widetilde{\nabla}$ becomes a linear connection on E when P is the identity morphism on E.

The above operator $\widetilde{\nabla}_X$ can be extended to $F(M)$–linear mappings $N : (\Gamma(E))^r \longrightarrow \Gamma(E)$ for any positive integer r. In particular, for the identity morphism I_E on E we have

$$(\widetilde{\nabla}_X I_E)(S) = \widetilde{\nabla}_X P(S) - P(\widetilde{\nabla}_X S), \ \ \forall\, X \in \Gamma(TM), \ S \in \Gamma(E).$$

The curvature form $\widetilde{\Omega}$ of $\widetilde{\nabla}$ is defined as follows (cf. Abe [Abe85])

$$\begin{aligned}\widetilde{\Omega}(X,Y)S = \widetilde{\nabla}_X\widetilde{\nabla}_Y P(S) - \widetilde{\nabla}_Y\widetilde{\nabla}_X P(S) - P(\widetilde{\nabla}_{[X,Y]}P(S))\\ -(\widetilde{\nabla}_X I_E)(\widetilde{\nabla}_Y S) + (\widetilde{\nabla}_Y I_E)(\widetilde{\nabla}_X S),\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $S \in \Gamma(E)$. For the particular case $E = TM$, an Otsuki connection $\widetilde{\nabla}$ has a torsion tensor field $\widetilde{T}$ given by (cf. Nemoto [Nem85])

$$\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - P([X,Y]), \ \ \forall\, X, Y \in \Gamma(TM). \tag{2.22}$$

Now, we show that starting with a linear connection ∇ on a vector bundle E we can obtain an Otsuki connection $\widetilde{\nabla}$ on a vector bundle G that is larger than E and $\widetilde{\nabla} = \nabla$ on E. Indeed, suppose $G = E \oplus F$, where F is another vector bundle over M, and denote by P the projection morphism of G on E. Then for any $X \in \Gamma(TM)$ we define the differential operator

$$\widetilde{\nabla}_X : \Gamma(G) \longrightarrow \Gamma(G); \ \widetilde{\nabla}_X S = \nabla_X P(S), \ \ \forall\, S \in \Gamma(G). \tag{2.23}$$

It is easy to check that $\widetilde{\nabla}$ is an Otsuki connection on G with respect to the vector bundle morphism P and $\widetilde{\nabla} = \nabla$ on E. Moreover, the following has been proved.

Theorem 2.8. (Bejancu–Otsuki [BO87]). *The restriction of the curvature form $\widetilde{\Omega}$ of $\widetilde{\nabla}$ to the sections of E coincides with the curvature form Ω of ∇.*

Next, we apply the theory of Otsuki connections to the study of an almost product manifold $(M, \mathcal{D}, \mathcal{D}')$. First, suppose that ∇ is a linear connection on $\mathcal{D}$ and consider the Otsuki connection $\widetilde{\nabla}$ on TM with respect to the decomposition (1.9) such that $\widetilde{\nabla} = \nabla$ on $\mathcal{D}$. Then according to (2.23) we have

$$\widetilde{\nabla}_X Y = \nabla_X QY, \ \ \forall\, X, Y \in \Gamma(TM). \tag{2.24}$$

Taking into account the relationship between the curvature forms of $\widetilde{\nabla}$ and ∇ stated in Theorem 2.8, we define a torsion tensor field T of ∇ as the restriction of the torsion tensor field $\widetilde{T}$ of $\widetilde{\nabla}$ to $\Gamma(TM)\times\Gamma(\mathcal{D})$. It is noteworthy that T is $\Gamma(\mathcal{D})$–valued. More precisely, by using (2.22) and (2.24) we obtain

$$T(X,QY) = \widetilde{T}(X,QY) = \nabla_X QY - \nabla_{QY} QX - Q[X,QY], \tag{2.25}$$

for any $X, Y \in \Gamma(TM)$. As T depends on $\mathcal{D}'$ we call it the $\mathcal{D}'$**–torsion tensor field** of ∇. Similarly, a linear connection ∇' on $\mathcal{D}'$ has a $\mathcal{D}$**–torsion tensor field** T' given by

$$T'(X,Q'Y) = \nabla'_X Q'Y - \nabla'_{Q'Y} Q'X - Q'[X,Q'Y], \quad \forall\, X,Y \in \Gamma(TM). \tag{2.26}$$

Finally, with respect to a non–holonomic frame field $\{E_A\}$ on $\mathcal{U} \subset M$ we put:

$$\text{(a) } T(E_j, E_i) = T_i{}^k{}_j E_k, \qquad \text{(b) } T(E_\alpha, E_i) = T_i{}^k{}_\alpha E_k, \tag{2.27}$$

and

$$\text{(a) } T'(E_\gamma, E_\alpha) = T'{}_\alpha{}^\beta{}_\gamma E_\beta, \qquad \text{(b) } T'(E_i, E_\alpha) = T'{}_\alpha{}^\beta{}_i E_\beta. \tag{2.28}$$

Then by using (2.7), (2.8), (2.12), (2.13) and (2.25)–(2.28) we deduce the local components of T and T' with respect to $\{E_A\}$ as they are expressed in the next theorem.

Theorem 2.9. *Let ∇ and ∇' be linear connections on the complementary distributions $\mathcal{D}$ and $\mathcal{D}'$ on M. Then the local components of T and T' with respect to the non–holonomic frame field $\{E_A\}$ are given by*

$$\begin{aligned} &\text{(a) } T_i{}^k{}_j = \Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i - V_i{}^k{}_j, \\ &\text{(b) } T_i{}^k{}_\alpha = \Gamma_i{}^k{}_\alpha - V_i{}^k{}_\alpha, \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} &\text{(a) } T'{}_\alpha{}^\beta{}_\gamma = \Gamma'{}_\alpha{}^\beta{}_\gamma - \Gamma'{}_\gamma{}^\beta{}_\alpha - V'{}_\alpha{}^\beta{}_\gamma, \\ &\text{(b) } T'{}_\alpha{}^\beta{}_i = \Gamma'{}_\alpha{}^\beta{}_i - V'{}_\alpha{}^\beta{}_i, \end{aligned} \tag{2.30}$$

respectively.

As the pair (∇, ∇') defines an adapted linear connection ∇^* on M we should see what relationship exists (if any) between their torsion tensor fields. First, by (2.14), (2.25) and (2.26) we deduce that T and T' are not equal to the restrictions of T^* on $\Gamma(TM)\times\Gamma(\mathcal{D})$ and $\Gamma(TM)\times\Gamma(\mathcal{D}')$ respectively. However, comparing Theorems 2.5 and 2.9 we see that the local components of T and T' form a part of the local components of T^* with respect to a non–holonomic frame field $\{E_A\}$ on M.

1.3 The Schouten–Van Kampen and Vrănceanu Connections

In the first part of this section we study the existence of adapted linear connections on an almost product manifold $(M, \mathcal{D}, \mathcal{D}')$. More precisely, we construct two adapted linear connections which were first introduced by Schouten and Van Kampen [SVK30] and Vrănceanu [VG31] for studying non–holonomic manifolds. Then we determine the general form of all adapted linear connections on $(M, \mathcal{D}, \mathcal{D}')$ and present these two special connections in an invariant form.

As M is supposed to be paracompact, by a result stated in Brickell–Clark [BC70], p. 154, there exists a linear connection $\widetilde{\nabla}$ on M. Then, locally we set

$$\widetilde{\nabla}_{E_B} E_A = F_A{}^C{}_B E_C, \tag{3.1}$$

where $\{E_A\} = \{E_i, E_\alpha\}$ is a non–holonomic frame field on $\mathcal{U} \subset M$. By direct calculations using (3.1) with respect to two non–holonomic frame fields $\{E_A\}$ and $\{\widetilde{E}_A\}$ on $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ we obtain the following transformations of non–holonomic coefficients of $\widetilde{\nabla}$ on $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$:

$$\begin{aligned}
&\text{(a) } \widetilde{F}_s{}^h{}_t a_h^k = (F_i{}^k{}_j a_s^i + (a_s^k)_{\|j}) a_t^j, && \text{(b) } \widetilde{F}_s{}^\beta{}_t a_\beta^\alpha = F_i{}^\alpha{}_j a_s^i a_t^j, \\
&\text{(c) } \widetilde{F}_s{}^h{}_\gamma a_h^k = (F_i{}^k{}_\alpha a_s^i + (a_s^k)_{|\alpha}) a_\gamma^\alpha, && \text{(d) } \widetilde{F}_s{}^\beta{}_\gamma a_\beta^\alpha = F_i{}^\alpha{}_\varepsilon a_s^i a_\gamma^\varepsilon, \\
&\text{(e) } \widetilde{F}_\nu{}^\varepsilon{}_j a_\varepsilon^\beta = (F_\alpha{}^\beta{}_i a_\nu^\alpha + (a_\nu^\beta)_{\|i}) a_j^i, && \text{(f) } \widetilde{F}_\nu{}^h{}_j a_h^k = F_\alpha{}^k{}_i a_\nu^\alpha a_j^i, \\
&\text{(g) } \widetilde{F}_\nu{}^\varepsilon{}_\mu a_\varepsilon^\beta = (F_\alpha{}^\beta{}_\gamma a_\nu^\alpha + (a_\nu^\beta)_{|\gamma}) a_\mu^\gamma, && \text{(h) } \widetilde{F}_\nu{}^h{}_\mu a_h^k = F_\alpha{}^k{}_\beta a_\nu^\alpha a_\mu^\beta,
\end{aligned} \tag{3.2}$$

with respect to (1.15). From (3.2a), (3.2c), (3.2e) and (3.2g) we deduce that $(\Gamma_i{}^k{}_A, \Gamma_\alpha{}^\beta{}_A)$ given by

$$\text{(a)} \quad \Gamma_i{}^k{}_A = F_i{}^k{}_A, \qquad \text{(b)} \quad \Gamma_\alpha{}^\beta{}_A = F_\alpha{}^\beta{}_A, \tag{3.3}$$

satisfy the conditions of Corollary 2.4. Hence they define an adapted linear connection ∇° on M. With respect to this connection we have to note that the formulas (60) from the book of Vrănceanu [VG57], p.235, are the same as our (3.3). As these formulas were first obtained by Schouten and Van Kampen [SVK30], we call the adapted linear connection $\nabla^\circ = (\Gamma_i{}^k{}_A, \Gamma_\alpha{}^\beta{}_A)$ given by (3.3) the **Schouten–Van Kampen connection**.

In order to define another adapted linear connection on M we consider (2.12c) on $\widetilde{\mathcal{U}} \subset M$ and $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$. Then by using elementary properties of the Lie bracket and taking into account (1.15), (2.6a) and (2.12c) we obtain

$$\begin{aligned}
Q[\widetilde{E}_\gamma, \widetilde{E}_s] &= Q[a_\gamma^\alpha E_\alpha, a_s^i E_i] = a_\gamma^\alpha a_s^i Q[E_\alpha, E_i] + a_\gamma^\alpha (a_s^i)_{|\alpha} E_i \\
&= (V_i{}^k{}_\alpha a_s^i + (a_s^k)_{|\alpha}) a_\gamma^\alpha E_k.
\end{aligned}$$

On the other hand, by (2.12c) on $\widetilde{\mathcal{U}}$ and (1.15a) we have

$$Q[\widetilde{E}_\gamma, \widetilde{E}_s] = \widetilde{V}_s{}^h{}_\gamma \widetilde{E}_h = \widetilde{V}_s{}^h{}_\gamma a_h^k E_k.$$

Comparing these equalities we deduce that $V_i{}^k{}_\alpha$ satisfy (2.9b) with respect to (1.15). In a similar way it follows that $V'{}_\alpha{}^\beta{}_i$ satisfy (2.10a). Hence, according to (3.2a) and (3.2g) the functions $(\Gamma^*{}_i{}^k{}_A, \Gamma^*{}_\alpha{}^\beta{}_A)$ given on each $\mathcal{U} \subset M$ by

$$\begin{aligned} &(a)\ \Gamma^*{}_i{}^k{}_j = F_i{}^k{}_j, \quad (b)\ \Gamma^*{}_i{}^k{}_\alpha = V_i{}^k{}_\alpha, \\ &(c)\ \Gamma^*{}_\alpha{}^\beta{}_i = V'{}_\alpha{}^\beta{}_i, \ (d)\ \Gamma^*{}_\alpha{}^\beta{}_\gamma = F_\alpha{}^\beta{}_\gamma, \end{aligned} \tag{3.4}$$

also satisfy the conditions of Corollary 2.4. The above adapted linear connection was first introduced by Vrănceanu [VG31]. Indeed, it is easy to see that formulas (21) of Vrănceanu [VG31], p. 199, are the same as our (3.4). The same formulas can be found in the book of Vrănceanu [VG57] (see formulas (61) at p. 235). Thus we are entitled to call the adapted linear connection $\nabla^* = (\Gamma^*{}_i{}^k{}_A, \Gamma^*{}_\alpha{}^\beta{}_A)$ the **Vrănceanu connection**.

Next, consider the torsion tensor field $\widetilde{T}$ of $\widetilde{\nabla}$ and by using (3.1) and (2.12)–(2.14) we obtain its local components with respect to the non–holonomic frame field $\{E_A\}$:

$$\begin{aligned} &(a)\ \widetilde{T}_A{}^k{}_B = F_A{}^k{}_B - F_B{}^k{}_A - V_A{}^k{}_B, \\ &(b)\ \widetilde{T}_A{}^\alpha{}_B = F_A{}^\alpha{}_B - F_B{}^\alpha{}_A - V'{}_A{}^\alpha{}_B. \end{aligned} \tag{3.5}$$

Also, by using (3.3), (3.4) and Theorem 2.5 we obtain the following.

Theorem 3.1. *The local components of the torsion tensor fields T° and T^* of Schouten–Van Kampen and Vrănceanu connections with respect to the non–holonomic frame field $\{E_A\}$ are given by*

$$\begin{aligned} &(a)\ T_i{}^k{}_j = F_i{}^k{}_j - F_j{}^k{}_i - V_i{}^k{}_j, \quad (b)\ T_i{}^\alpha{}_j = -V'{}_i{}^\alpha{}_j, \\ &(c)\ T_i{}^k{}_\alpha = -T_\alpha{}^k{}_i = F_i{}^k{}_\alpha - V_i{}^k{}_\alpha, \ (d)\ T_\alpha{}^\beta{}_i = -T_i{}^\beta{}_\alpha = F_\alpha{}^\beta{}_i - V'{}_\alpha{}^\beta{}_i, \\ &(e)\ T_\alpha{}^k{}_\beta = -V_\alpha{}^k{}_\beta, \qquad (f)\ T_\alpha{}^\gamma{}_\beta = F_\alpha{}^\gamma{}_\beta - F_\beta{}^\gamma{}_\alpha - V'{}_\alpha{}^\gamma{}_\beta, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &(a)\ T^*{}_i{}^k{}_j = F_i{}^k{}_j - F_j{}^k{}_i - V_i{}^k{}_j, \ (b)\ T^*{}_i{}^\alpha{}_j = -V'{}_i{}^\alpha{}_j, \\ &(c)\ T^*{}_i{}^k{}_\alpha = -T^*{}_\alpha{}^k{}_i = 0, \qquad (d)\ T^*{}_\alpha{}^\beta{}_i = -T^*{}_i{}^\beta{}_\alpha = 0, \\ &(e)\ T^*{}_\alpha{}^k{}_\beta = -V_\alpha{}^k{}_\beta, \qquad (f)\ T^*{}_\alpha{}^\gamma{}_\beta = F_\alpha{}^\gamma{}_\beta - F_\beta{}^\gamma{}_\alpha - V'{}_\alpha{}^\gamma{}_\beta, \end{aligned} \tag{3.7}$$

respectively.

Corollary 3.2. *The Schouten–Van Kampen and Vrănceanu connections coincide if and only if they have the same torsion tensor fields.*

From (3.5)–(3.7) we see that even when $\widetilde{\nabla}$ is torsion–free, the Schouten–Van Kampen and Vrănceanu connections are not necessarily torsion–free. Related to this, by using (3.5) and (3.7) we obtain the following.

Theorem 3.3. (Vrănceanu [VG57], p. 235). *Let $\widetilde{\nabla}$ be a torsion–free linear connection on $(M, \mathcal{D}, \mathcal{D}')$. Then the Vrănceanu connection determined by $\widetilde{\nabla}$ is torsion–free if and only if both distributions $\mathcal{D}$ and $\mathcal{D}'$ are involutive.*

This made Vrănceanu ([VG57], p. 236) remark that the connection ∇^* is more intimately related to the properties of the manifold than ∇°. This remark will become more evident as we go further into the study of non–holonomic semi–Riemannian manifolds and semi–Riemannian foliations.

According to Theorem 2.6 we may write down all the local components of curvature tensor fields of ∇° and ∇^* with respect to a non–holonomic frame field. However, since for ∇° we just replace Γ and Γ' from (2.20) and (2.21) by F, we omit them here. We only apply Theorem 2.6 for ∇^* and obtain the following.

Theorem 3.4. *The local components of the curvature tensor field of the Vrănceanu connection ∇^* with respect to a non–holonomic frame field $\{E_A\}$ are given by*

$$\begin{aligned}
&\text{(a)}\ R^*{}_i{}^h{}_{jk} = F_i{}^h{}_{j\|k} - F_i{}^h{}_{k\|j} + F_i{}^s{}_j F_s{}^h{}_k - F_i{}^s{}_k F_s{}^h{}_j \\
&\qquad\qquad - F_i{}^h{}_s V_j{}^s{}_k - V_i{}^h{}_\alpha V'{}_j{}^\alpha{}_k, \\
&\text{(b)}\ R^*{}_i{}^h{}_{\alpha k} = V_i{}^h{}_{\alpha\|k} - F_i{}^h{}_{k|\alpha} + V_i{}^s{}_\alpha F_s{}^h{}_k - F_i{}^s{}_k V_s{}^h{}_\alpha \\
&\qquad\qquad - F_i{}^h{}_s V_\alpha{}^s{}_k - V_i{}^h{}_\varepsilon V'{}_\alpha{}^\varepsilon{}_k, \\
&\text{(c)}\ R^*{}_i{}^h{}_{\alpha\beta} = V_i{}^h{}_{\alpha|\beta} - V_i{}^h{}_{\beta|\alpha} + V_i{}^s{}_\alpha V_s{}^h{}_\beta - V_i{}^s{}_\beta V_s{}^h{}_\alpha \\
&\qquad\qquad - F_i{}^h{}_s V_\alpha{}^s{}_\beta - V_i{}^h{}_\varepsilon V'{}_\alpha{}^\varepsilon{}_\beta,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\text{(a)}\ R^*{}_\alpha{}^\beta{}_{jk} = V'{}_\alpha{}^\beta{}_{j\|k} - V'{}_\alpha{}^\beta{}_{k\|j} + V'{}_\alpha{}^\varepsilon{}_j V'{}_\varepsilon{}^\beta{}_k - V'{}_\alpha{}^\varepsilon{}_k V'{}_\varepsilon{}^\beta{}_j \\
&\qquad\qquad - V'{}_\alpha{}^\beta{}_s V_j{}^s{}_k - F_\alpha{}^\beta{}_\varepsilon V'{}_j{}^\varepsilon{}_k, \\
&\text{(b)}\ R^*{}_\alpha{}^\beta{}_{\gamma k} = F_\alpha{}^\beta{}_{\gamma\|k} - V'{}_\alpha{}^\beta{}_{k|\gamma} + F_\alpha{}^\varepsilon{}_\gamma V'{}_\varepsilon{}^\beta{}_k - V'{}_\alpha{}^\varepsilon{}_k F_\varepsilon{}^\beta{}_\gamma \\
&\qquad\qquad - V'{}_\alpha{}^\beta{}_s V_\gamma{}^s{}_k - F_\alpha{}^\beta{}_\varepsilon V'{}_\gamma{}^\varepsilon{}_k, \\
&\text{(c)}\ R^*{}_\alpha{}^\beta{}_{\gamma\mu} = F_\alpha{}^\beta{}_{\gamma|\mu} - F_\alpha{}^\beta{}_{\mu|\gamma} + F_\alpha{}^\varepsilon{}_\gamma F_\varepsilon{}^\beta{}_\mu - F_\alpha{}^\varepsilon{}_\mu F_\varepsilon{}^\beta{}_\gamma \\
&\qquad\qquad - V'{}_\alpha{}^\beta{}_s V_\gamma{}^s{}_\mu - F_\alpha{}^\beta{}_\varepsilon V'{}_\gamma{}^\varepsilon{}_\mu.
\end{aligned} \tag{3.9}$$

Now, we want to express the general form of all adapted linear connections on $(M, \mathcal{D}, \mathcal{D}')$ and then to describe the Schouten–Van Kampen and Vrănceanu connections in an invariant form. First we prove the following general result.

Theorem 3.5. *Let $(M, \mathcal{D}, \mathcal{D}')$ be an almost product manifold and $\widetilde{\nabla}$ be a linear connection on M. Then all the adapted linear connections on M are given by*

$$\nabla_X Y = Q\widetilde{\nabla}_X QY + Q'\widetilde{\nabla}_X Q'Y + QS(X, QY) + Q'S(X, Q'Y), \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$*, where* S *is an arbitrary tensor field of type* $(1,2)$ *on* M*.*

Proof. It is easy to check that ∇ given by (3.10) is an adapted linear connection on M. Conversely, suppose that ∇ is an adapted linear connection on M. Then we put

$$\nabla_X Y - \widetilde{\nabla}_X Y = S(X,Y), \ \forall X, Y \in \Gamma(TM), \tag{3.11}$$

where S is a tensor field of type $(1,2)$ on M. Next, by using (2.1) and (2.2), we have

$$Q'(\nabla_X QY) = 0 \text{ and } Q(\nabla_X Q'Y) = 0, \ \ \forall X, Y \in \Gamma(TM).$$

Thus by (3.11) we deduce that

$$Q'(\widetilde{\nabla}_X QY + S(X, QY)) = 0 \text{ and } Q(\widetilde{\nabla}_X Q'Y + S(X, Q'Y)) = 0, \tag{3.12}$$

for any $X, Y \in \Gamma(TM)$. Finally, by using (3.12) in (3.11) we obtain (3.10). ∎

Next, we define:

$$S^{\circ}(X,Y) = Q'\widetilde{\nabla}_X QY + Q\widetilde{\nabla}_X Q'Y,$$

and

$$S^{*}(X,Y) = Q([Q'X, QY] - \widetilde{\nabla}_{Q'X} QY) + Q'([QX, Q'Y] - \widetilde{\nabla}_{QX} Q'Y),$$

for any $X, Y \in \Gamma(TM)$. It is easy to check that both S° and S^{*} are tensor fields of type $(1,2)$ on M. Then, by direct calculations we deduce that

$$\text{(a) } QS^{\circ}(X, QY) = 0, \quad \text{(b) } \ Q'S^{\circ}(X, Q'Y) = 0, \tag{3.13}$$

and

$$\begin{aligned} &\text{(a) } QS^{*}(X, QY) = Q([Q'X, QY] - \widetilde{\nabla}_{Q'X} QY), \\ &\text{(b) } Q'S^{*}(X, Q'Y) = Q'([QX, Q'Y] - \widetilde{\nabla}_{QX} Q'Y), \end{aligned} \tag{3.14}$$

for any $X, Y \in \Gamma(TM)$. Finally, by using in turn (3.13) and (3.14) in the general form (3.10) we obtain two adapted linear connections ∇° and ∇^{*} given by

$$\nabla^{\circ}_X Y = Q\widetilde{\nabla}_X QY + Q'\widetilde{\nabla}_X Q'Y, \tag{3.15}$$

and

$$\nabla^{*}_X Y = Q\widetilde{\nabla}_{QX} QY + Q'\widetilde{\nabla}_{Q'X} Q'Y + Q[Q'X, QY] + Q'[QX, Q'Y], \tag{3.16}$$

for any $X, Y \in \Gamma(TM)$. Moreover, we prove the following theorem.

Theorem 3.6. *The adapted linear connections given by* (3.15) *and* (3.16) *are the Schouten–Van Kampen and Vrănceanu connections respectively.*

Proof. Replace the pair (X, Y) from (3.15) and (3.16) in turn by (E_j, E_i), (E_α, E_i), (E_i, E_α) and (E_γ, E_α) and using (3.1), (2.12) and (2.13) we obtain the local coefficients of Schouten–Van Kampen and Vrănceanu connections given by (3.3) and (3.4) respectively. ■

The coordinate–free forms (3.15) and (3.16) of Schouten–Van Kampen and Vrănceanu connections were first obtained by Ianuş [Ian71] and then used by Băditoiu, Buchner and Ianuş [BBI98] for studying semi–Riemannian submersions.

1.4 From Semi–Euclidean Algebra to Semi–Riemannian Geometry

For the sake of completeness of the book, and to present our terminology, we start with some basic notions and results about semi–Euclidean spaces.

Let V be a real $m-$dimensional vector space and $g : V \times V \longrightarrow \mathbb{R}$ be a symmetric bilinear mapping. We say that g is a **scalar product** on V if it is **non–degenerate**, that is, whenever $g(u, v) = 0$ for all $v \in V$, then $u = 0$. The vector space V endowed with a scalar product g is denoted by (V, g) and it is called a **semi–Euclidean** (**pseudo–Euclidean**) **space**. Let q be the dimension of the largest subspace W of (V, g) on which g is negative definite, i.e., $g(w, w) < 0$ for any non-zero vector $w \in W$. Then we say that g is of **index** q. When $q = 0$ (resp. $q = 1$), (V, g) is called a **Euclidean space** (resp. **Lorentz** (**Minkowski**) **space**). If $0 < q < m$, then we say that (V, g) is a **proper semi–Euclidean space**. In such a vector space we have three categories of vectors as follows. A vector $v \in V$ is called:

$$\textbf{space–like}\ , \text{ if } g(v, v) > 0 \quad \text{or} \quad v = 0,$$
$$\textbf{light–like (null)}\ , \text{ if } g(v, v) = 0 \ \text{ and } v \neq 0,$$
$$\textbf{time–like}\ , \text{ if } g(v, v) < 0.$$

The **length** (**norm**) of $v \in V$ is the non–negative number $\|v\| = |g(v, v)|^{1/2}$. When $\|v\| = 1$ we say that v is a **unit vector**. Two vectors v and w are **orthogonal** if $g(v, w) = 0$. Contrary to the case of Euclidean geometry, a light–like vector of a proper semi–Euclidean space is a non-zero vector that is orthogonal to itself. A basis of (V, g) formed by m mutually orthogonal unit vectors is called an **orthonormal basis**. The existence of such bases is ensured by the following.

Lemma 4.1. (O'Neill [O83], p. 50).

Lemma 4.1. (O'Neill [O83], p. 50).

(i) *Any semi–Euclidean space* (V, g) *with* $V \neq \{0\}$ *has an orthonormal basis* $B = \{e_1, ..., e_m\}$.
(ii) *Any vector* $v \in V$ *has a unique expression*

$$v = \sum_{i=1}^{m} \varepsilon_i g(v, e_i) e_i,$$

where $\varepsilon_i = g(e_i, e_i)$.

Next, we consider a subspace W of a semi–Euclidean space (V, g). Then the restriction of g to W is a symmetric bilinear form on W which we also denote by g. If g is non–degenerate on W, then (W, g) is also a semi–Euclidean space. Any subspace $W \neq \{0\}$ of a Euclidean space (V, g) is a Euclidean space too. However, when (V, g) is a proper semi–Euclidean space g might be **degenerate** on W, that is, there exists a non-zero vector $u \in W$ such that

$$g(u, w) = 0, \quad \text{for all } w \in W. \tag{4.1}$$

When g is degenerate (resp. non–degenerate) on a subspace W of (V, g) we say that W is a **degenerate** (resp. **non–degenerate**) **subspace** of (V, g).

Lemma 4.2. *Any* $m-$ *dimensional proper semi–Euclidean space with* $m \geq 2$ *has both degenerate and non–degenerate subspaces.*

Proof. According to (i) of Lemma 4.1 we consider an orthonormal basis B of (V, g). If $u \in B$, then $W = \text{span}\{u\}$ is a non–degenerate subspace of (V, g). Since g is of index $0 < q < m$, there exist in B at least one time–like vector u and one space–like vector v. Then $W' = \text{span}\{u + v\}$ is a degenerate subspace of (V, g). ■

To discuss the degree of degeneracy of a subspace W we define the **orthogonal subspace** $W^\perp$ to W in (V, g) by

$$W^\perp = \{u \in V : g(u, w) = 0, \ \forall\, w \in W\}. \tag{4.2}$$

In general, $W^\perp$ is not complementary to W in V, but the following equalities are true:

$$\dim W + \dim W^\perp = m, \tag{4.3}$$

and

$$(W^\perp)^\perp = W. \tag{4.4}$$

Moreover, we have the following.

Lemma 4.3. W *is a non–degenerate subspace of a semi–Euclidean space if and only if* $W^\perp$ *is non–degenerate too.*

Proof. Suppose W is non–degenerate and $W^\perp$ is degenerate. Then there exists $u \in W^\perp$, $u \neq 0$, such that

$$g(u, w^\perp) = 0, \quad \text{for all } w^\perp \in W^\perp. \tag{4.5}$$

On the other hand, by the definition of $W^\perp$ we have

$$g(u, w) = 0, \quad \text{for all } w \in W. \tag{4.6}$$

From (4.5) and (4.4) it follows that $u \in W$. Then by (4.6) we deduce that W is degenerate, which is a contradiction. Thus $W^\perp$ must be non–degenerate. Conversely, if $W^\perp$ is non–degenerate, then by the above reason we infer that $(W^\perp)^\perp$ is non–degenerate. Hence by (4.4), W is non–degenerate. ∎

Corollary 4.4. *W is a degenerate subspace of a semi–Euclidean space (V, g) if and only if $W^\perp$ is degenerate too.*

Now, we consider the **null subspace** of $W \subset (V, g)$ with respect to g, denoted by $\mathcal{N}(W, g)$ and defined by

$$\mathcal{N}(W, g) = \{u \in W : g(u, w) = 0, \quad \forall\, w \in W\}. \tag{4.7}$$

By using (4.4) and (4.7) we deduce that

$$\mathcal{N}(W, g) = \mathcal{N}(W^\perp, g) = W \cap W^\perp. \tag{4.8}$$

The dimension of the null subspace of W is called the **nullity degree** of W with respect to g, and it is denoted by *null* (W, g). Then the following can be easily proved.

Lemma 4.5. *Let (V, g) be a semi–Euclidean space and W be a subspace of V. Then we have the assertions:*

(i) *W is a degenerate subspace of (V, g) if and only if null $(W, g) > 0$.*
(ii) *W is a non–degenerate subspace of (V, g) if and only if null $(W, g) = 0$.*

Let *null*$(W, g) = r$. If $r > 0$ we say that (W, g) is an r–**degenerate subspace** of (V, g). According to Walker [Wal50a] the n–dimensional subspace (W, g) is called:

partially–null , if $0 < r < n$,

totally–null , if $r = n$,

non–null , if $r = 0$.

Lemma 4.6. *Let (W, g) be a partially–null subspace of a semi–Euclidean space (V, g). Then any complementary subspace to $\mathcal{N}(W, g)$ in W is non–degenerate.*

Proof. Let $S(W, g)$ be a complementary subspace to $\mathcal{N}(W, g)$ in W. Suppose that $S(W, g)$ is degenerate. Then there exists a non–zero vector $v \in S(W, g)$

such that $g(v, w) = 0$ for any $w \in S(W, g)$. As $v \in W$ and $\mathcal{N}(W, g)$ is the null subspace of W, we have $g(v, u) = 0$, for any $u \in \mathcal{N}(W, g)$. Thus v is orthogonal to all vectors of W and hence it is a vector in $\mathcal{N}(W, g)$. This is a contradiction because $v \neq 0$ and $\mathcal{N}(W, g)$ and $S(W, g)$ are complementary subspaces of W. Therefore, $S(W, g)$ must be non–degenerate. ∎

A complementary subspace to $\mathcal{N}(W, g)$ in a partially–null subspace W of (V, g) is called a **screen subspace**. Later on, in Sections 1.8 and 3.5 we shall see that screen subspaces are fibers of some distributions which play an important role in studying degenerate distributions (resp. foliations).

Finally, we define the **light–like** (**null**) **cone** of a proper semi–Euclidean space (V, g) as the set Λ of all light–like vectors in V, that is, we have

$$\Lambda = \{v \in V \setminus \{0\} : g(v, v) = 0\}.$$

Clearly Λ is not a subspace of V, but it contains $\mathcal{N}(W, g) \setminus \{0\}$ for any degenerate subspace W of (V, g).

Example 4.1. Let $\mathbb{R}^m$ be the space of m–tuples $(x^1, ..., x^m) = x$ of real numbers. For any $0 < q < m$ define $g : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ by

$$g(x, y) = -\sum_{t=1}^{q} x^t y^t + \sum_{s=q+1}^{m} x^s y^s. \tag{4.9}$$

Then $\mathbb{R}^m_q = (\mathbb{R}^m, g)$ is a proper semi–Euclidean space of index q. In particular, $\mathbb{R}^m_1$ is a Lorentz (Minkowski) vector space with g given by

$$g(x, y) = -x^1 y^1 + \sum_{s=2}^{m} x^s y^s. \tag{4.10}$$

Finally, $\mathbb{R}^m$ becomes a Euclidean space with respect to the usual inner product

$$g(x, y) = \sum_{s=1}^{m} x^s y^s. \tag{4.11}$$

∎

Example 4.2. Consider in $\mathbb{R}^4_1$ the subspaces:

$$\begin{aligned} W &= \{x \in \mathbb{R}^4 : x^1 + x^2 + x^3 + x^4 = 0\}, \\ W' &= \{x \in \mathbb{R}^4 : x^1 = x^2\}, \\ W'' &= \{x \in \mathbb{R}^4 : x^1 = x^2,\ x^3 = x^4 = 0\}. \end{aligned}$$

Then it is easy to see that W, W' and W'' are non–null, partially–null and totally–null subspaces of $\mathbb{R}^4_1$, respectively. Moreover, we have

$$\mathcal{N}(W', g) = W'',$$

and

$$\Lambda \cap W' = W'' \setminus \{0\},$$

where Λ is the light–like cone of $\mathbb{R}^4_1$, i.e., we have

$$\Lambda = \{x \in \mathbb{R}^4 \setminus \{0\} : -(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 0\}. \qquad \blacksquare$$

Now, we extend the above concepts to distributions and manifolds. Let M be an $(n+p)$– dimensional manifold endowed with an n–distribution $\mathcal{D}$. Denote by $L^2_s(\mathcal{D}_x, \mathbb{R})$ the real vector space of all symmetric bilinear mappings $g_x : \mathcal{D}_x \times \mathcal{D}_x \longrightarrow \mathbb{R}$. Then we consider the vector bundle

$$L^2_s(\mathcal{D}, \mathbb{R}) = \bigcup_{x \in M} L^2_s(\mathcal{D}_x, \mathbb{R}),$$

over M, and give the following definition. A **semi–Riemannian metric** of index q on $\mathcal{D}$ is a smooth section $g : x \longrightarrow g_x$ of $L^2_s(\mathcal{D}, \mathbb{R})$ such that each g_x is non–degenerate of index q on $\mathcal{D}_x$ for all $x \in M$. When $q = 0$, that is g_x is positive definite for any $x \in M$, we say that g is a **Riemannian metric** on $\mathcal{D}$. According to this terminology for the metric, we say that $(\mathcal{D}, g)$ is a **semi–Riemannian distribution** of index q, and when $q = 0$ it is a **Riemannian distribution**. Finally, if $q = 1$ we say that $(\mathcal{D}, g)$ is a **Lorentz distribution**. We note that if not stated otherwise, the theory is developed regardless of the integrability of $\mathcal{D}$.

If in particular $\mathcal{D} = TM$, then g becomes a **semi–Riemannian metric** on M and (M, g) is called a **semi–Riemannian (pseudo–Riemannian) manifold** (cf. O'Neill, [O83], p. 54) of index q. In case $q = 0$ (resp. $q = 1$), (M, g) is said to be a **Riemannian manifold** (resp. **Lorentz manifold**). When $0 < q < n+p$ we call (M, g) a **proper semi–Riemannian manifold**. In this case each pair (T_xM, g_x) is a proper semi–Euclidean space of constant index q.

We discuss next the non–linear counter–part of the algebraic study considered in the first part of this section. This takes us to the theory of non–holonomic manifolds as substructures of semi–Riemannian manifolds. More precisely, we consider an n–distribution $\mathcal{D}$ on an $(n+p)$–dimensional semi–Riemannian manifold (M, g). Then g induces a global section of $L^2_s(\mathcal{D}, \mathbb{R})$ which we denote by the same symbol g. Two important cases will be considered in our study. One is when $(\mathcal{D}, g)$ is a semi–Riemannian distribution on M and the other one is when each $\mathcal{D}_x$ is an r–degenerate subspace of the semi–Euclidean space (T_xM, g_x) for all $x \in M$. In the latter case we say that $(\mathcal{D}, g)$ is an r–**degenerate distribution** on M. This case occurs only when (M, g) is a proper semi–Riemannian manifold. When (M, g) is a Riemannian manifold then any $(\mathcal{D}, g)$ is a Riemannian distribution.

Now, let (M, g) be a semi–Riemannian manifold and $(\mathcal{D}, g)$ be a semi–Riemannian distribution on M. Then we consider the vector bundle

$$\mathcal{D}^{\perp} = \bigcup_{x \in M} \mathcal{D}_x^{\perp},$$

where $\mathcal{D}_x^{\perp}$ is the complementary orthogonal subspace to $\mathcal{D}_x$ in (T_xM, g_x). By Lemma 4.3 we deduce that g induces a semi–Riemannian metric g' on $\mathcal{D}^{\perp}$, and therefore $(\mathcal{D}^{\perp}, g')$ is a semi–Riemannian distribution too. Thus in this case we may consider $\mathcal{D}^{\perp}$ as transversal distribution to $\mathcal{D}$ and study this geometric structure by using some natural linear connections (cf. Sections 1.5, 1.6, 1.7). When $(\mathcal{D}, g)$ is an $r-$degenerate $n-$distribution, the construction of a transversal distribution seems to be more difficult to achieve. When $r < n$ (resp. $r = n$), $\mathcal{D}_x$ is a partially–null (resp. totally–null) subspace of T_xM for any $x \in M$, so we call $\mathcal{D}$ a **partially–null** (resp. **totally–null**) **distribution** on (M, g).

1.5 Intrinsic and Induced Linear Connections on Semi–Riemannian Distributions

Let M be an $(n+p)-$dimensional manifold and $\mathcal{D}$ be an n–distribution on M. Suppose g is a semi–Riemannian metric on $\mathcal{D}$, that is, $(\mathcal{D}, g)$ is a semi–Riemannian distribution on M. First we want to construct a linear connection on $\mathcal{D}$ whose properties are very similar to those of the Levi–Civita connection on a semi–Riemannian manifold. To this end we consider a complementary distribution $\mathcal{D}'$ to $\mathcal{D}$ in TM. Then a linear connection ∇ on $\mathcal{D}$ is said to be $\mathcal{D}'$**–torsion–free** if its $\mathcal{D}'$–torsion tensor field T vanishes on M, i.e., by (2.25) we have

$$\nabla_X QY - \nabla_{QY} QX - Q[X, QY] = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{5.1}$$

Also, we say that g is $\mathcal{D}$**–parallel** (**parallel along** $\mathcal{D}$) with respect to ∇ if we have

$$\begin{aligned}(\nabla_{QX}\, g)(QY, QZ) = {} & QX(g(QY, QZ)) - g(\nabla_{QX} QY, QZ) \\ & - g(QY, \nabla_{QX} QZ) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM).\end{aligned} \tag{5.2}$$

Now, we can state the following theorem.

Theorem 5.1. (Bejancu–Farran [BF05]). *Let $(\mathcal{D}, g)$ be a semi–Riemannian distribution on M and $\mathcal{D}'$ be a complementary distribution to $\mathcal{D}$ in TM. Then there exists a unique linear connection D on $\mathcal{D}$ satisfying the following conditions:*

(i) *D is $\mathcal{D}'$–torsion–free.*
(ii) *g is $\mathcal{D}$–parallel with respect to D.*

Proof. Define the differential operator $D : \Gamma(TM) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D})$ by

$$\begin{aligned} 2g(D_{QX}QY, QZ) = & \, QX(g(QY, QZ)) + QY(g(QZ, QX)) \\ & - QZ(g(QX, QY)) + g(Q[QX, QY], QZ) \\ & - g(Q[QY, QZ], QX) + g(Q[QZ, QX], QY), \end{aligned} \tag{5.3}$$

and

$$D_{Q'X}QY = Q[Q'X, QY], \tag{5.4}$$

for any $X, Y, Z \in \Gamma(TM)$. It is easy to verify that D given by (5.3) and (5.4) is a linear connection on $\mathcal{D}$ that satisfies the conditions (i) and (ii). Next, suppose that ∇ is another linear connection on $\mathcal{D}$ satisfying (i) and (ii). Since ∇ is $\mathcal{D}'$–torsion–free, from (5.1) we deduce that

$$\nabla_{Q'X}QY = Q[Q'X, QY], \tag{5.5}$$

and

$$\nabla_{QX}QY - \nabla_{QY}QX - Q[QX, QY] = 0, \tag{5.6}$$

for any $X, Y \in \Gamma(TM)$. Now, by using (5.2) and (5.6) we obtain

$$\begin{aligned} 0 & = (\nabla_{QX}\, g)(QY, QZ) + (\nabla_{QY} g)(QZ, QX) - (\nabla_{QZ} g)(QX, QY) \\ & = QX(g(QY, QZ)) + QY(g(QZ, QX)) - QZ(g(QX, QY)) \\ & + g(Q[QX, QY], QZ) - g(Q[QY, QZ], QX) + g(Q[QZ, QX], QY) \\ & - 2g(\nabla_{QX}QY, QZ). \end{aligned} \tag{5.7}$$

Finally, comparing (5.5) and (5.7) with (5.4) and (5.3) respectively, we conclude that $\nabla = D$, which proves the uniqueness of D. ■

In general, a linear connection $\widetilde{\nabla}$ on a manifold M is called **torsion–free** if its torsion tensor field vanishes, that is, we have (see (2.14))

$$\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y] = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{5.8}$$

If (M, g) is a semi–Riemannian manifold then we say that g is **parallel** with respect to $\widetilde{\nabla}$ if we have

$$\begin{aligned} (\widetilde{\nabla}_X g)(Y, Z) = & \, X(g(Y, Z)) - g(\widetilde{\nabla}_X Y, Z) \\ & - g(Y, \widetilde{\nabla}_X Z) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM). \end{aligned} \tag{5.9}$$

Corollary 5.2. *Let (M, g) be a semi–Riemannian manifold. Then there exists a unique linear connection $\widetilde{\nabla}$ on M satisfying the following conditions:*

(i) *$\widetilde{\nabla}$ is torsion–free.*

(ii) *g is parallel with respect to $\widetilde{\nabla}$.*

Proof. Apply Theorem 5.1 for the case $\mathcal{D} = TM$. Then we have only the trivial complementary distribution $\mathcal{D}' = \{0\}$ and thus $Q = I$ and $Q' = 0$. Hence (5.1) and (5.2) become (5.8) and (5.9) respectively. Finally, (5.3) gives the linear connection we are looking for, that is,

$$\begin{aligned} 2g(\widetilde{\widetilde{\nabla}}_X Y, Z) = {} & X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ & + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y), \end{aligned} \tag{5.10}$$

for any $X, Y, Z \in \Gamma(TM)$. ■

The linear connection $\widetilde{\widetilde{\nabla}}$ given by (5.10) is the well known **Levi–Civita connection** which was considered as a miracle of semi–Riemannian geometry (cf. O'Neill [O83], p. 60).

The local coefficients of $\widetilde{\widetilde{\nabla}}$ with respect to the natural frame field $\left\{ \dfrac{\partial}{\partial x^a} \right\}$ on M can be easily obtained from (5.10). To achieve this we put:

$$\text{(a)} \quad \widetilde{\widetilde{\nabla}}_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^a} = \left\{ {c \atop a \;\; b} \right\} \frac{\partial}{\partial x^c}, \quad \text{(b)} \quad g_{ab} = g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right). \tag{5.11}$$

Then replace X, Y and Z from (5.10) by $\dfrac{\partial}{\partial x^b}$, $\dfrac{\partial}{\partial x^a}$ and $\dfrac{\partial}{\partial x^d}$ respectively, and by using (5.11) and taking into account that $\left[\dfrac{\partial}{\partial x^a}, \dfrac{\partial}{\partial x^b} \right] = 0$ for any $a, b \in \{1, ..., n+p\}$, we obtain

$$2 \left\{ {c \atop a \;\; b} \right\} g_{cd} = \frac{\partial g_{ad}}{\partial x^b} + \frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d}.$$

Finally, we deduce that the well known **Christoffel coefficients** $\left\{ {c \atop a \;\; b} \right\}$ for the Levi–Civita connection on M are given by

$$\left\{ {c \atop a \;\; b} \right\} = \frac{1}{2} g^{cd} \left(\frac{\partial g_{ad}}{\partial x^b} + \frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d} \right), \tag{5.12}$$

where g^{cd} are the entries of the inverse matrix of $[g_{cd}]$. When $\mathcal{D}$ is integrable we will get in Section 3.1 the local coefficients for the linear connection D.

Next we consider an $(n+p)$–dimensional semi–Riemannian manifold (M, g) and suppose that $(\mathcal{D}, g)$ is a semi–Riemannian n–distribution on M. Then $(\mathcal{D}^{\perp}, g)$ is a semi–Riemannian p–distribution on M. Here we denoted by the same symbol g the semi–Riemannian metrics induced by g on $\mathcal{D}$ and $\mathcal{D}^{\perp}$. Thus we have

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}. \tag{5.13}$$

According to Theorem 5.1 there exists a unique connection D (resp. $D^{\perp}$) on $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) satisfying the conditions from the theorem with respect to the decomposition (5.13). We call D and $D^{\perp}$ the **intrinsic connections** on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. In what follows we keep the same notations Q and Q' for the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

Theorem 5.3. *The adapted linear connection determined by the pair of intrinsic connections* $(D, D^{\perp})$ *is just the Vrănceanu connection* ∇^* *defined by the Levi–Civita connection* $\widetilde{\nabla}$ *on* (M, g).

Proof. First, by using (5.3), (5.4) and (5.10) we deduce that

$$D_X QY = Q\widetilde{\nabla}_{QX} QY + Q[Q'X, QY], \quad \forall\, X, Y \in \Gamma(TM), \tag{5.14}$$

where $\widetilde{\nabla}$ is the Levi–Civita connection on (M, g). Similarly, it follows that

$$D^{\perp}_X Q'Y = Q'\widetilde{\nabla}_{Q'X} Q'Y + Q'[QX, Q'Y], \quad \forall\, X, Y \in \Gamma(TM). \tag{5.15}$$

Then we use (5.14), (5.15) and (2.4) and obtain (3.16), which proves that the Vrănceanu connection ∇^* defined by $\widetilde{\nabla}$ is the adapted connection determined by $(D, D^{\perp})$. ■

Next by using (2.14) and (2.4) for ∇^*, D and $D^{\perp}$ we deduce that

$$Q(T^*(X, QY)) = D_X QY - D_{QY} QX - Q[X, QY],$$

and

$$Q'(T^*(X, Q'Y)) = D^{\perp}_X Q'Y - D^{\perp}_{Q'Y} Q'X - Q'[X, Q'Y],$$

for any $X, Y \in \Gamma(TM)$. These formulas together with (5.1), (5.2) and Theorems 5.1 and 5.3 enable us to state the following corollary.

Corollary 5.4. *The Vrănceanu connection* ∇^* *defined by the Levi–Civita connection* $\widetilde{\nabla}$ *on* (M, g) *is the only adapted linear connection on* M *satisfying the following conditions*

$$\begin{array}{ll} \text{(a) } (\nabla^*_{QX} g)(QY, QZ) = 0, & \text{(b) } (\nabla^*_{Q'X} g)(Q'Y, Q'Z) = 0, \\ \text{(c) } Q(T^*(X, QY)) = 0, & \text{(d) } Q'(T^*(X, Q'Y)) = 0, \end{array} \tag{5.16}$$

for any $X, Y, Z \in \Gamma(TM)$.

On the other hand, the Levi–Civita connection $\widetilde{\nabla}$ on (M, g) induces some linear connections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$. Thus it is interesting to see if these connections coincide with the intrinsic connections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$. We show that this happens if and only if M is a locally semi–Riemannian product manifold.

First, according to (5.13) we write

$$\widetilde{\nabla}_X QY = \nabla_X QY + h(X, QY), \tag{5.17}$$

and

$$\widetilde{\nabla}_X Q'Y = h'(X, Q'Y) + \nabla^{\perp}_X Q'Y, \tag{5.18}$$

where we set:

$$\text{(a)}\ \ \nabla_X QY = Q\widetilde{\nabla}_X QY, \qquad \text{(b)}\ \ \nabla^{\perp}_X Q'Y = Q'\widetilde{\nabla}_X Q'Y, \tag{5.19}$$

and

$$\text{(a)}\ \ h(X, QY) = Q'\widetilde{\nabla}_X QY, \qquad \text{(b)}\ \ h'(X, Q'Y) = Q\widetilde{\nabla}_X Q'Y, \tag{5.20}$$

for any $X, Y \in \Gamma(TM)$. We call (5.17) and (5.18) the **Gauss formulas** for the semi–Riemannian distributions $(\mathcal{D}, g)$ and $(\mathcal{D}^{\perp}, g)$ respectively. It is easy to check that ∇ and $\nabla^{\perp}$ are linear connections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively, while h and h' are $F(M)$–bilinear mappings:

$$h : \Gamma(TM)\times\Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}^{\perp}), \quad h' : \Gamma(TM)\times\Gamma(\mathcal{D}^{\perp}) \longrightarrow \Gamma(\mathcal{D}).$$

We call ∇ (resp. $\nabla^{\perp}$) the **induced connection** by $\widetilde{\nabla}$ on $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$). Also, we call

$$h : \Gamma(\mathcal{D})\times\Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}^{\perp}) \ \text{ and } \ h' : \Gamma(\mathcal{D}^{\perp})\times\Gamma(\mathcal{D}^{\perp}) \longrightarrow \Gamma(\mathcal{D}),$$

given by

$$\begin{aligned} &\text{(a)}\ h(QX, QY) = Q'\widetilde{\nabla}_{QX} QY \quad \text{and} \\ &\text{(b)}\ h'(Q'X, Q'Y) = Q\widetilde{\nabla}_{Q'X} Q'Y, \end{aligned} \tag{5.21}$$

the **second fundamental forms** of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. Next, for any $Q'X \in \Gamma(\mathcal{D}^{\perp})$ and $QX \in \Gamma(\mathcal{D})$ we define the $F(M)$–linear operators

$$A_{Q'X} : \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}) \ \text{ and } \ A'_{QX} : \Gamma(\mathcal{D}^{\perp}) \longrightarrow \Gamma(\mathcal{D}^{\perp}),$$

by

$$\begin{aligned} &\text{(a)}\ A_{Q'X} QY = -h'(QY, Q'X) \quad \text{and} \\ &\text{(b)}\ A'_{QX} Q'Y = -h(Q'Y, QX). \end{aligned} \tag{5.22}$$

According to the theory of submanifolds, we call $A_{Q'X}$ and A'_{QX} the **shape operators** of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to the normal sections $Q'X$ and QX respectively. By using (5.9), (5.17) and (5.18) we obtain

$$g(h(X, QY), Q'Z) + g(h'(X, Q'Z), QY) = 0, \ \ \forall\, X, Y, Z \in \Gamma(TM). \tag{5.23}$$

As a consequence of (5.21)–(5.23) we deduce that the second fundamental forms and the shape operators of the distributions $\mathcal{D}$ and $\mathcal{D}'$ are related by

$$g(h(QX, QY), Q'Z) = g(A_{Q'Z} QX, QY), \tag{5.24}$$

and

$$g(h'(Q'X, Q'Y), QZ) = g(A'_{QZ}Q'X, Q'Y). \tag{5.25}$$

Finally, from (5.17) and (5.18) we infer that

$$\begin{aligned} &\text{(a) } \widetilde{\nabla}_{QX}QY = \nabla_{QX}QY + h(QX, QY), \\ &\text{(b) } \widetilde{\nabla}_{Q'X}QY = \nabla_{Q'X}QY - A'_{QY}Q'X, \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} &\text{(a) } \widetilde{\nabla}_{QX}Q'Y = -A_{Q'Y}QX + \nabla^{\perp}_{QX}Q'Y, \\ &\text{(b) } \widetilde{\nabla}_{Q'X}Q'Y = h'(Q'X, Q'Y) + \nabla^{\perp}_{Q'X}Q'Y. \end{aligned} \tag{5.27}$$

As from now on we refer only to the decomposition (5.13) dictated by the semi–Riemannian metric g, we call the $\mathcal{D}^{\perp}$ (resp. $\mathcal{D}$)–torsion tensor field of ∇ (resp. $\nabla^{\perp}$) simply **torsion tensor field**. Now, we say that g is **parallel** with respect to a linear connection ∇' on $\mathcal{D}$, if we have

$$\begin{aligned} (\nabla'_X g)(QY, QY) = X(g(QY, QZ)) - g(\nabla'_X QY, QZ) \\ - g(QY, \nabla'_X QZ) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM). \end{aligned} \tag{5.28}$$

Then we prove the following.

Lemma 5.5.

(i) *The semi–Riemannian metric g on $\mathcal{D}$ is parallel with respect to the induced connection ∇.*

(ii) *The torsion tensor field of ∇ is given by*

$$T(X, QY) = \nabla_{Q'X}QY - Q[Q'X, QY], \quad \forall\, X, Y \in \Gamma(TM). \tag{5.29}$$

(iii) *$\mathcal{D}$ is an involutive distribution if and only if one of the following two conditions is satisfied:*
  (a) *The second fundamental form h of $\mathcal{D}$ is symmetric.*
  (b) *The shape operator $A_{Q'Z}$ of $\mathcal{D}$ is symmetric with respect to g for any $Q'Z \in \Gamma(\mathcal{D}^{\perp})$.*

Proof. The assertion (i) follows from (5.9) by using (5.19a) and (5.28) for ∇. Next by using (5.19a) in (2.25) and taking into account (5.8) we obtain (5.29). Finally, (5.21a) and (5.8) imply

$$h(QX, QY) - h(QY, QX) = Q'[QX, QY], \quad \forall\, X, Y \in \Gamma(TM),$$

which proves that $\mathcal{D}$ is involutive if and only if the second fundamental form of $\mathcal{D}$ is symmetric. The equivalence of (iiia) and (iiib) is a consequence of (5.24). ∎

We note that when $\mathcal{D}$ is an integrable distribution then h defined by (5.21a) determines the second fundamental form for any local leaf M^* of $\mathcal{D}$. Recall from the theory of submanifolds that M^* is **totally geodesic** at a point $x \in M^*$, if for every $v \in T_xM^*$ the geodesic $x^a = x^a(t)$ of M determined by (x, v) lies in M^* for small values of the parameter t. If M^* is totally geodesic at every point, then it is called a **totally geodesic submanifold** of M. It is proved that M^* is totally geodesic if and only if its second fundamental form vanishes identically on M^* (cf. O'Neill [O83], p. 104).

Now, from (5.29) we deduce that the induced connection ∇ on $\mathcal{D}$, in general, is not torsion–free, so it does not coincide with the intrinsic connection D on $\mathcal{D}$. The following theorem sheds more light on this issue.

Theorem 5.6. *Let $(\mathcal{D}, g)$ be a semi–Riemannian distribution on the semi–Riemannian manifold (M, g). Then the following assertions are equivalent:*

(i) *The induced connection ∇ coincides with the intrinsic connection D on $\mathcal{D}$.*
(ii) *The second fundamental form h of $\mathcal{D}$ vanishes identically on M.*
(iii) *$\mathcal{D}$ is integrable and its local leaves are totally geodesic immersed in (M, g).*

Proof. By Theorem 5.1 we know that D is the only linear connection on $\mathcal{D}$ which is torsion–free and with respect to which g is $\mathcal{D}$–parallel. Taking into account that g is also $\mathcal{D}$–parallel with respect to ∇ (cf. (i) of Lemma 5.5), and by using (5.29) and (5.19a) we deduce that $\nabla = D$ if and only if

$$Q\widetilde{\nabla}_{Q'X}QY = Q[Q'X, QY], \quad \forall\, X, Y \in \Gamma(TM). \tag{5.30}$$

Next, by using (5.10) we compute $2g(Q\widetilde{\nabla}_{Q'X}QY, QZ)$ and infer that (5.30) is equivalent to

$$\begin{aligned} Q'X(g(QY, QZ)) - g([Q'X, QY], QZ) - g([QY, QZ], Q'X) \\ + g([QZ, Q'X], QY) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM). \end{aligned} \tag{5.31}$$

By using (5.8) and (5.9) we deduce that (5.31) is equivalent to

$$g(\widetilde{\nabla}_{QY}QZ, Q'X) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM). \tag{5.32}$$

From (5.32), by using (5.20a), we obtain that $\nabla = D$ if and only if

$$h(QY, QZ) = 0, \quad \forall\, Y, Z \in \Gamma(TM), \tag{5.33}$$

which proves the equivalence of (i) and (ii). Finally, by using the assertion (iiia) of Lemma 5.5 we deduce that (5.33) is satisfied if and only if $\mathcal{D}$ is integrable and its local leaves are totally geodesic immersed in (M, g). This proves the equivalence of (ii) and (iii) and completes the proof of our theorem. ∎

Theorem 5.7. *The adapted linear connection determined by the pair of induced connections $(\nabla, \nabla^{\perp})$ is just the Schouten–Van Kampen connection ∇° defined by the Levi–Civita connection $\widetilde{\nabla}$ on (M, g).*

Proof. The assertion follows by using the coordinate–free form (3.15) of the Schouten–Van Kampen connection and (5.19). ∎

Now we define two classes of manifolds that are going to be studied in detail in Chapter 4. When both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are integrable, we say that M is a **locally product manifold**. If moreover, the local leaves of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are totally geodesic immersed in (M, g) then we say that M is a **locally semi–Riemannian product**.

Next, we note that Theorem 5.6 is also true for $(\mathcal{D}^{\perp}, g)$. Then taking into account Theorems 5.6, 5.3 and 5.7 we obtain the following interesting characterization of locally semi–Riemannian products.

Theorem 5.8. *Let $(\mathcal{D}, g)$ and $(\mathcal{D}^{\perp}, g)$ be two complementary orthogonal semi–Riemannian distributions on the semi–Riemannian manifold (M, g). Then M is a locally semi–Riemannian product with respect to the decomposition* (5.13) *if and only if the Schouten–Van Kampen and Vrănceanu connections defined by the Levi–Civita connection on (M, g) coincide.*

As, in general, the second fundamental form h of $\mathcal{D}$ is not symmetric (cf. assertion (iii) of Lemma 5.5) we define the **symmetric second fundamental form** h^s of $\mathcal{D}$ by

$$h^s(QX, QY) = \frac{1}{2}(h(QX, QY) + h(QY, QX)), \quad \forall X, Y \in \Gamma(TM). \tag{5.34}$$

Also, we say that a vector field X on M is a $\mathcal{D}$–**Killing vector field** if

$$(\mathcal{L}_X\, g)(QY, QZ) = g(\widetilde{\nabla}_{QY}X, QZ) + g(\widetilde{\nabla}_{QZ}X, QY) = 0, \tag{5.35}$$

for any $Y, Z \in \Gamma(TM)$, where $\mathcal{L}$ is the Lie derivative on M.

Now, we remark that, in general, g is not parallel with respect to any of the intrinsic connections D and $D^{\perp}$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. More precisely, we have

Theorem 5.9. *Let $(\mathcal{D}, g)$ be a semi–Riemannian distribution on the semi–Riemannian manifold (M, g). Then the following assertions are equivalent:*

(i) *g is parallel with respect to the intrinsic connection D on $\mathcal{D}$.*
(ii) *$Q'X$ is a $\mathcal{D}$–Killing vector field, for any $X \in \Gamma(TM)$.*
(iii) *The symmetric second fundamental form of $\mathcal{D}$ vanishes identically on M.*

Proof. Since g is $\mathcal{D}$–parallel with respect to D (see (ii) of Theorem 5.1), we deduce that g is parallel with respect to D if and only if it is $\mathcal{D}^{\perp}$–parallel with respect to D, that is,

$$(D_{Q'X}\, g)(QY, QZ) = 0, \quad \forall X, Y, Z \in \Gamma(TM). \tag{5.36}$$

Then by using (5.4), (5.8) and (5.9) we infer that (5.36) is equivalent to

$$\begin{aligned}0 &= Q'X(g(QY,QZ)) - g([Q'X,QY],QZ) - g(QY,[Q'X,QZ]) \\ &= g(\widetilde{\nabla}_{Q'X}QY,QZ) + g(QY,\widetilde{\nabla}_{Q'X}QZ) - g(\widetilde{\nabla}_{Q'X}QY,QZ) \\ &+ g(\widetilde{\nabla}_{QY}Q'X,QZ) - g(QY,\widetilde{\nabla}_{Q'X}QZ) + g(QY,\widetilde{\nabla}_{QZ}Q'X) \\ &= g(\widetilde{\nabla}_{QY}Q'X,QZ) + g(QY,\widetilde{\nabla}_{QZ}Q'X).\end{aligned} \tag{5.37}$$

Thus by (5.37) and (5.35) we obtain the equivalence of (i) and (ii). Finally, by using (5.9), (5.21a) and (5.34) we deduce that (5.37) is equivalent to

$$0 = g(Q'X, \widetilde{\nabla}_{QY}QZ + \widetilde{\nabla}_{QZ}QY) = 2g(Q'X, h^s(QY,QZ)),$$

which completes the proof of the theorem. ■

So far we have obtained characterizations of two important classes of distributions on (M,g). More precisely, one class concerns semi–Riemannian distributions $(\mathcal{D},g)$ for which $\nabla = D$. The second deals with semi–Riemannian distributions for which g is parallel with respect to the intrinsic connection. These two classes can be related as follows.

Theorem 5.10. *Let $(\mathcal{D},g)$ be a semi–Riemannian distribution on the semi–Riemannian manifold (M,g). Then the following assertions are equivalent:*

(i) *The induced connection ∇ coincides with the intrinsic connection D on $\mathcal{D}$.*
(ii) *The induced connection ∇ on $\mathcal{D}$ is torsion–free.*
(iii) *$\mathcal{D}$ is integrable and g is parallel with respect to D.*

Proof. (i) $\Longrightarrow$ (ii). As D is torsion–free, it follows that ∇ must be torsion–free too. (ii) $\Longrightarrow$ (i). Since ∇ is torsion–free and g is parallel with respect to ∇ (cf. (i) of Lemma 5.5), by uniqueness of D stated by Theorem 5.1 we obtain $\nabla = D$. (i) $\Longleftrightarrow$ (iii). By assertion (iiia) of Lemma 5.5 and Theorem 5.9 we deduce that the assertion (iii) of the theorem holds if and only if the second fundamental form h of $\mathcal{D}$ vanishes identically on M. Then apply Theorem 5.6 and obtain the equivalence of (i) and (iii). ■

Next, by direct calculations using (2.14) and (5.29) for both $\mathcal{D}$ and $\mathcal{D}^\perp$ we deduce that

$$T^\circ(Q'X,QY) = T(X,QY) - T^\perp(Y,Q'X), \ \forall X,Y \in \Gamma(TM), \tag{5.38}$$

where T°, T and $T^\perp$ are the torsion tensor fields of ∇°, ∇ and $\nabla^\perp$ respectively. Moreover, by using (2.14), (3.15) and (5.8) we obtain

$$T^\circ(QX,QY) = T^\circ(Q'X,Q'Y) = 0, \ \forall X,Y \in \Gamma(TM). \tag{5.39}$$

Theorem 5.11. *Let ∇° be the Schouten–Van Kampen connection defined by the Levi–Civita connection $\widetilde{\nabla}$ on (M,g) with respect to the decomposition* (5.13). *Then the following assertions are equivalent:*

(i) ∇° *coincides with* $\widetilde{\nabla}$.
(ii) ∇° *is torsion–free.*
(iii) *Both induced connections are torsion–free.*

Proof. (i)$\Longrightarrow$(ii). As $\widetilde{\nabla}$ is torsion–free, it follows that ∇° is torsion–free too. (ii)$\Longrightarrow$(i). Since g is parallel with respect to both ∇ and $\nabla^\perp$ (cf. (i) of Lemma 5.5) we have

$$(\nabla^\circ_X g)(QY, QZ) = (\nabla^\circ_X g)(Q'Y, Q'Z) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM).$$

Also, taking into account that ∇° is an adapted connection to the decomposition (5.13) we obtain

$$(\nabla^\circ_X g)(Q'Y, QZ) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM).$$

Hence ∇° satisfies both (5.8) and (5.9) and by uniqueness of $\widetilde{\nabla}$ from Corollary 5.2 we must have $\nabla^\circ = \widetilde{\nabla}$. Finally, the equivalence of (ii) and (iii) follows from (5.38) and (5.39). ■

Taking into account Theorems 5.8, 5.10 and 5.11 we state the following.

Theorem 5.12. *Let* $(\mathcal{D}, g)$ *and* $(\mathcal{D}^\perp, g)$ *be two complementary orthogonal semi–Riemannian distributions on the semi–Riemannian manifold* (M, g)*. If* $\widetilde{\nabla}, \nabla^\circ$ *and* ∇^* *represent the Levi–Civita, Schouten–Van Kampen and Vrănceanu connections respectively, then the following assertions are equivalent:*

(i) *M is a locally semi–Riemannian product with respect to the decomposition* (5.13).
(ii) $\nabla^\circ = \nabla^*$.
(iii) $\nabla^\circ = \widetilde{\nabla}$.
(iv) $\nabla^* = \widetilde{\nabla}$.

From Theorem 5.10 we see that the condition $\nabla = D$ on $\mathcal{D}$ is stronger than the condition for g being parallel with respect to D. The latter condition was first introduced by Reinhart [Rei59a] for foliated manifolds, that is, $\mathcal{D}^\perp$ is supposed to be an integrable distribution. A Riemannian (semi–Riemannian) metric satisfying this condition was called a **bundle–like metric** and it was intensively studied by several authors (see Tondeur [Ton97] for references, and Section 3.3 for more details). It is interesting to see whether bundle–like metrics can be found on a non–holonomic semi–Riemannian manifold $(M, g, \mathcal{D}, \mathcal{D}^\perp)$, that is, when none of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ is integrable. The next example shows that the answer is in the affirmative.

Example 5.1. Let $(\mathbb{R}^4, g)$ be the 4–dimensional Euclidean space with g given by (4.11) for $m = 4$. We define the open submanifold M of $\mathbb{R}^4$ by

$$M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : 2x^3 - (x^1)^2 > 0\},$$

where (x^1, x^2, x^3, x^4) is a rectangular coordinate system on $\mathbb{R}^4$. Then on the Riemannian manifold (M, g) we consider the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ spanned by

$$\left\{X_1 = \frac{\partial}{\partial x^1} + L\,\frac{\partial}{\partial x^2} + x^1\,\frac{\partial}{\partial x^3},\ X_2 = \frac{\partial}{\partial x^4} + x^1\,\frac{\partial}{\partial x^2} - L\,\frac{\partial}{\partial x^3}\right\},$$

and

$$\left\{Y_1 = \frac{\partial}{\partial x^2} - L\,\frac{\partial}{\partial x^1} - x^1\,\frac{\partial}{\partial x^4},\ Y_2 = \frac{\partial}{\partial x^3} - x^1\,\frac{\partial}{\partial x^1} + L\,\frac{\partial}{\partial x^4}\right\},$$

respectively, where $L = \sqrt{2x^3 - (x^1)^2}$. It is easy to see that $\mathcal{D}$ and $\mathcal{D}^\perp$ are complementary orthogonal Riemannian distributions on (M, g). Moreover, none of them is involutive, so by Frobenius Theorem (see Theorem 2.1.7) they are not integrable. However, we show that g is parallel with respect to the intrinsic connection D on $\mathcal{D}$. To this end we first note that we should verify only (5.36). Taking into account that $\{X_1, X_2\}$ is an orthogonal basis in $\Gamma(\mathcal{D})$, from the first equality in (5.37) we deduce that g is parallel with respect to D if and only if

$$g([Y_i, X_1], X_2) + g([Y_i, X_2], X_1) = 0, \quad i \in \{1, 2\}.$$

By direct calculations it follows that these equalities are satisfied and hence g is a bundle–like metric on M. Thus this is an example of a bundle–like metric on a Riemannian manifold (M, g) endowed with two complementary orthogonal non–integrable distributions. ∎

1.6 Fundamental Equations for Semi–Riemannian Distributions

Let (M, g) be a semi–Riemannian manifold endowed with two complementary othogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$. In the previous section we constructed the intrinsic connections D and $D^\perp$ on $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively and proved that the pair $(D, D^\perp)$ determines the Vrănceanu connection ∇^* (cf. Theorem 5.3). Also, the Levi–Civita connection $\widetilde{\nabla}$ on (M, g) induces two linear connections ∇ and $\nabla^\perp$ on $\mathcal{D}$ and $\mathcal{D}^\perp$. Moreover, the pair $(\nabla, \nabla^\perp)$ determines the Schouten–Van Kampen connection ∇° (cf. Theorem 5.7).

In the present section we first relate the curvature tensor of $\widetilde{\nabla}$ to the curvature tensors of $\nabla, \nabla^\perp$ and ∇°. Then we obtain equations connecting curvature tensors of the Schouten–Van Kampen and Vrănceanu connections. As a consequence we deduce the equations which relate the curvature tensors of $\widetilde{\nabla}$ and ∇^*.

The theory we develop here is done in the general situation when none of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ is supposed to be integrable. First, by using the linear connections $\nabla, \nabla^{\perp}$ and ∇° we define the covariant derivatives of h and h' given by (5.20) as follows:

$$(\nabla^{\perp}_X h)(Y, QZ) = \nabla^{\perp}_X (h(Y, QZ)) - h(\nabla^{\circ}_X Y, QZ) - h(Y, \nabla_X QZ), \quad (6.1)$$

$$(\nabla_X h')(Y, Q'Z) = \nabla_X (h'(Y, Q'Z)) - h'(\nabla^{\circ}_X Y, Q'Z) - h(Y, \nabla'_X Q'Z), \quad (6.2)$$

for any $X, Y, Z \in \Gamma(TM)$. Denote by $\widetilde{R}, R$ and $R^{\perp}$ the curvature tensors of the linear connections $\widetilde{\nabla}, \nabla$ and $\nabla^{\perp}$ respectively, and by T° the torsion tensor field of the Schouten–Van Kampen connection ∇°. Then we state the following.

Theorem 6.1. *Let $(\mathcal{D}, g)$ and $(\mathcal{D}^{\perp}, g)$ be two complementary orthogonal semi–Riemannian distributions on the semi–Riemannian manifold (M, g). Then we have the following equations:*

(i) *$\mathcal{D}$–Gauss Equation:*

$$\begin{aligned} g(\widetilde{R}(X,Y)QZ, QU) &= g(R(X,Y)QZ, QU) + g(h(X,QZ), h(Y,QU)) \\ &\quad - g(h(Y,QZ), h(X,QU)), \end{aligned} \quad (6.3)$$

(ii) *$\mathcal{D}$–Codazzi–Equation:*

$$\begin{aligned} g(\widetilde{R}(X,Y)QZ, Q'U) &= g((\nabla^{\perp}_X h)(Y,QZ) - (\nabla^{\perp}_Y h)(X,QZ), Q'U) \\ &\quad + g(h(T^{\circ}(X,Y), QZ), Q'U), \end{aligned} \quad (6.4)$$

(iii) *$\mathcal{D}^{\perp}$–Gauss Equation:*

$$\begin{aligned} g(\widetilde{R}(X,Y)Q'Z,Q'U) &= g(R^{\perp}(X,Y)Q'Z,Q'U) + g(h'(X,Q'Z), h'(Y,Q'U)) \\ &\quad - g(h'(Y,Q'Z), h'(X,Q'U)), \end{aligned} \quad (6.5)$$

(iv) *$\mathcal{D}^{\perp}$–Codazzi Equation:*

$$\begin{aligned} g(\widetilde{R}(X,Y)Q'Z, QU) &= g((\nabla_X h')(Y,Q'Z) - (\nabla_Y h')(X,Q'Z), QU) \\ &\quad + g(h'(T^{\circ}(X,Y), Q'Z), QU), \end{aligned} \quad (6.6)$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Proof. By using the Gauss formulas (5.17) and (5.18) we deduce that

$$\begin{aligned} \widetilde{\nabla}_X \widetilde{\nabla}_Y QZ &= \nabla_X \nabla_Y QZ + h(X, \nabla_Y QZ) + h'(X, h(Y, QZ)) \\ &\quad + \nabla^{\perp}_X (h(Y, QZ)). \end{aligned} \quad (6.7)$$

On the other hand, (5.17) and (2.14) for ∇° imply

$$\begin{aligned} \widetilde{\nabla}_{[X,Y]} QZ &= \nabla_{[X,Y]} QZ + h(\nabla^{\circ}_X Y, QZ) - h(\nabla^{\circ}_Y X, QZ) \\ &\quad - h(T^{\circ}(X,Y), QZ). \end{aligned} \quad (6.8)$$

Then using (6.7), (6.8) and (6.1) we obtain

$$\begin{aligned}\widetilde{R}(X,Y)QZ &= [\widetilde{\nabla}_X,\widetilde{\nabla}_Y]QZ - \widetilde{\nabla}_{[X,Y]}QZ \\ &= \{R(X,Y)QZ + h'(X,h(Y,QZ)) \\ &- h'(Y,h(X,QZ))\} + \{(\nabla^{\perp}_X h)(Y,QZ) \\ &- (\nabla^{\perp}_Y h)(X,QZ) + h(T^{\circ}(X,Y),QZ)\}.\end{aligned} \tag{6.9}$$

Taking the $\mathcal{D}^{\perp}$ – and $\mathcal{D}$– components in (6.9) we obtain (6.4) and

$$\begin{aligned}g(\widetilde{R}(X,Y)QZ,QU) &= g(R(X,Y)QZ,QU) + g(h'(X,h(Y,QZ)),QU) \\ &- g(h'(Y,h(X,QZ)),QU).\end{aligned} \tag{6.10}$$

Finally, we use (5.23) in (6.10) and obtain (6.3). In a similar way we deduce that

$$\begin{aligned}\widetilde{R}(X,Y)Q'Z &= \{(\nabla_X h')(Y,Q'Z) - (\nabla_Y h')(X,Q'Z) \\ &+ h'(T^{\circ}(X,Y),Q'Z)\} + \{R^{\perp}(X,Y)Q'Z \\ &+ h(X,h'(Y,Q'Z)) - h(Y,h'(X,Q'Z))\}.\end{aligned} \tag{6.11}$$

Then (6.5) and (6.6) follow from (6.11) by taking the $\mathcal{D}^{\perp}$– and $\mathcal{D}$–components respectively. ■

We call (6.3)–(6.6) the **fundamental equations** of the pair of distributions $(\mathcal{D},\mathcal{D}^{\perp})$ on (M,g). We note that (6.4) and (6.6) are equivalent to each other. To see this we first prove the following lemma.

Lemma 6.2.

(i) *The covariant derivatives of h and h' are related by*

$$g((\nabla^{\perp}_X h)(Y,QZ),Q'U) + g((\nabla_X h')(Y,Q'U),QZ) = 0. \tag{6.12}$$

(ii) *The torsion tensor field T° of Schouten–Van Kampen connection is given by*

$$\begin{aligned}T^{\circ}(X,Y) &= \{h'(Y,Q'X) - h'(X,Q'Y)\} \\ &+ \{h(Y,QX) - h(X,QY)\}.\end{aligned} \tag{6.13}$$

(iii) *The torsion tensor field T^* of Vrănceanu connection is given by*

$$\begin{aligned}T^*(X,Y) &= \{h'(Q'Y,Q'X) - h'(Q'X,Q'Y)\} \\ &+ \{h(QY,QX) - h(QX,QY)\}.\end{aligned} \tag{6.14}$$

Proof. The assertion (i) follows by direct calculations using (5.23) and taking into account that g is parallel with respect to both ∇ and $\nabla^{\perp}$ (cf. (i) of Lemma 5.5). Next, by using (5.17)–(5.20) and (3.15) we deduce that

$$\widetilde{\nabla}_X Y = \nabla^{\circ}_X Y + h(X, QY) + h'(X, Q'Y), \ \forall\, X, Y \in \Gamma(TM). \tag{6.15}$$

Then (5.8) and (6.15) imply

$$\begin{aligned} T^{\circ}(X,Y) &= (\nabla^{\circ}_X Y - \widetilde{\nabla}_X Y) - (\nabla^{\circ}_Y X - \widetilde{\nabla}_Y X) \\ &= h(Y, QX) + h'(Y, Q'X) - h(X, QY) - h'(X, Q'Y), \end{aligned}$$

which proves (6.13). Finally, by using (3.15), (3.16), (5.20) and (5.8) we obtain

$$\nabla^{\circ}_X Y = \nabla^{*}_X Y + h(Q'Y, QX) + h'(QY, Q'X), \ \forall\, X, Y \in \Gamma(TM). \tag{6.16}$$

Then by using (6.13) and (6.16) we obtain (6.14). ∎

Now, as a consequence of (5.23) we obtain

$$g(h(T^{\circ}(X,Y), QZ), Q'U) + g(h'(T^{\circ}(X,Y), Q'U), QZ) = 0. \tag{6.17}$$

Then, using (6.12) and (6.17) in the right part of (6.4) and taking into account that $\widetilde{R}$ satisfies

$$g(\widetilde{R}(X,Y)QZ, Q'U) + g(\widetilde{R}(X,Y)Q'U, QZ) = 0,$$

we deduce the equivalence of (6.4) and (6.6).

Theorem 6.3. *The curvature tensor fields $\widetilde{R}$ and R° of the Levi–Civita connection $\widetilde{\nabla}$ and of the Schouten–Van Kampen connection ∇° are related by*

$$\begin{aligned} \widetilde{R}(X,Y)Z = R^{\circ}(X,Y)Z &+ h'(X, h(Y, QZ)) - h'(Y, h(X, QZ)) \\ &+ h'(T^{\circ}(X,Y), Q'Z) + h(X, h'(Y, Q'Z)) \\ &- h(Y, h'(X, Q'Z)) + h(T^{\circ}(X,Y), QZ) \\ &+ (\nabla_X h')(Y, Q'Z) - (\nabla_Y h')(X, Q'Z) \\ &+ (\nabla^{\perp}_X h)(Y, QZ) - (\nabla^{\perp}_Y h)(X, QZ), \end{aligned} \tag{6.18}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. As ∇° is the adapted connection determined by the pair $(\nabla, \nabla^{\perp})$ (cf. Theorem 5.7), by using (2.4) we deduce that

$$\nabla^{\circ}_X Y = \nabla_X QY + \nabla^{\perp}_X Q'Y, \ \ \forall\, X, Y \in \Gamma(TM). \tag{6.19}$$

This implies

$$R^{\circ}(X,Y)Z = R(X,Y)QZ + R^{\perp}(X,Y)Q'Z, \quad \forall\, X,Y,Z \in \Gamma(TM). \tag{6.20}$$

Then adding (6.9) and (6.11), and taking into account (6.20) we obtain (6.18). ■

Next, by using the Vrănceanu connection we define the following covariant derivatives for h and h':

$$(\nabla^*_X h)(Z,QY) = \nabla^*_X(h(Z,QY)) - h(\nabla^*_X Z,QY) - h(Z,\nabla^*_X QY), \tag{6.21}$$

$$(\nabla^*_X h')(Z,Q'Y) = \nabla^*_X(h'(Z,Q'Y)) - h'(\nabla^*_X Z,Q'Y) - h(Z,\nabla^*_X Q'Y), \tag{6.22}$$

for any $X,Y,Z \in \Gamma(TM)$. Then we denote by R^* the curvature tensor field of the Vrănceanu connection ∇^* defined by the Levi–Civita connection $\widetilde{\nabla}$ on (M,g) and prove the following.

Theorem 6.4. *The curvature tensor fields R and $R^{\perp}$ of the induced connections ∇ and $\nabla^{\perp}$ are related to R^* by the following equations:*

$$\begin{aligned} R(X,Y)QZ = R^*(X,Y)QZ &+ (\nabla^*_X h')(QZ,Q'Y) \\ &- (\nabla^*_Y h')(QZ,Q'X) + h'(h'(QZ,Q'Y),Q'X) \\ &- h'(h'(QZ,Q'X),Q'Y) + h'(QZ,Q'T^*(X,Y)), \end{aligned} \tag{6.23}$$

and

$$\begin{aligned} R^{\perp}(X,Y)Q'Z = R^*(X,Y)Q'Z &+ (\nabla^*_X h)(Q'Z,QY) \\ &- (\nabla^*_Y h)(Q'Z,QX) + h(h(Q'Z,QY),QX) \\ &- h(h(Q'Z,QX),QY) + h(Q'Z,QT^*(X,Y)), \end{aligned} \tag{6.24}$$

for any $X,Y,Z \in \Gamma(TM)$.

Proof. By using (5.19a), (5.20b), (5.8) and (3.16) we deduce that

$$\nabla_Y QZ = \nabla^*_Y QZ + h'(QZ,Q'Y), \quad \forall\, Y,Z \in \Gamma(TM). \tag{6.25}$$

Then by direct calculations using (6.25) we obtain

$$\begin{aligned} R(X,Y)QZ = R^*(X,Y)QZ &+ h'(\nabla^*_Y QZ,Q'X) \\ &- h'(\nabla^*_X QZ,Q'Y) + \nabla^*_X(h'(QZ,Q'Y)) \\ &- \nabla^*_Y(h'(QZ,Q'X)) + h'(h'(QZ,Q'Y),Q'X) \\ &- h'(h'(QZ,Q'X),Q'Y) - h'(QZ,Q'[X,Y]). \end{aligned} \tag{6.26}$$

Using (2.14) and taking into account that ∇^* is an adapted linear connection on M with respect to the decomposition (5.13) (see (iii) of Theorem 2.2), we infer that

$$\begin{aligned} Q'[X,Y] &= Q'(\nabla^*_X Y) - Q'(\nabla^*_Y X) - Q'(T^*(X,Y)) \\ &= \nabla^*_X Q'Y - \nabla^*_Y Q'X - Q'(T^*(X,Y)). \end{aligned} \tag{6.27}$$

Now, we use (6.27) in the last term of (6.26) and via (6.22) we obtain (6.23). In a similar way (6.24) follows. ∎

By adding (6.23) and (6.24) and then using (6.20) we obtain the following corollary.

Corollary 6.5. *The curvature tensors R° and R^* of the Schouten–Van Kampen and Vrănceanu connections are related by*

$$\begin{aligned} R^\circ(X,Y)Z = {} & R^*(X,Y)Z + (\nabla^*_X h')(QZ, Q'Y) \\ & - (\nabla^*_Y h')(QZ, Q'X) + (\nabla^*_X h)(Q'Z, QY) \\ & - (\nabla^*_Y h)(Q'Z, QX) + h'(h'(QZ, Q'Y), Q'X) \\ & - h'(h'(QZ, Q'X), Q'Y) + h(h(Q'Z, QY), QX) \\ & - h(h(Q'Z, QX), QY) + h'(QZ, Q'T^*(X,Y)) \\ & + h(Q'Z, QT^*(X,Y)), \end{aligned} \tag{6.28}$$

for any $X, Y, Z \in \Gamma(TM)$.

Finally, combining Theorem 6.3 with Corollary 6.5 we state the following.

Corollary 6.6. *The curvature tensors $\widetilde{R}$ and R^* of Levi–Civita and Vrănceanu connections are related by*

$$\begin{aligned} \widetilde{R}(X,Y)Z = {} & R^*(X,Y)Z + (\nabla^*_X h')(QZ, Q'Y) \\ & -(\nabla^*_Y h')(QZ, Q'X) + (\nabla_X h')(Y, Q'Z) \\ & -(\nabla_Y h')(X, Q'Z) + (\nabla^*_X h)(Q'Z, QY) \\ & -(\nabla^*_Y h)(Q'Z, QX) + (\nabla^\perp_X h)(Y, QZ) \\ & -(\nabla^\perp_Y h)(X, QZ) + h'(h'(QZ, Q'Y), Q'X) \\ & -h'(h'(QZ, Q'X), Q'Y) + h'(X, h(Y, QZ)) \\ & -h'(Y, h(X, QZ)) + h'(QZ, Q'T^*(X,Y)) \\ & +h'(T^\circ(X,Y), Q'Z) + h(h(Q'Z, QY), QX) \\ & -h(h(Q'Z, QX), QY) + h(X, h'(Y, Q'Z)) \\ & -h(Y, h'(X, Q'Z)) + h(Q'Z, QT^*(X,Y)) \\ & +h(T^\circ(X,Y), QZ), \end{aligned} \tag{6.29}$$

for any $X, Y, Z \in \Gamma(TM)$.

We note that a different approach for studying semi–Riemannian distributions and submersions (see the definition in Section 2.1) was developed by Gray [Gra67] and O'Neill [O66]. Their study is based on two tensor fields T and A of type $(1,2)$ on M given by

$$T_XY = Q'\widetilde{\nabla}_{QX}QY + Q\widetilde{\nabla}_{QX}Q'Y, \tag{6.30}$$

and

$$A_XY = Q\widetilde{\nabla}_{Q'X}Q'Y + Q'\widetilde{\nabla}_{Q'X}QY, \tag{6.31}$$

for any $X, Y \in \Gamma(TM)$. By using (5.20) we deduce that

$$T_XY = h(QX, QY) + h'(QX, Q'Y), \tag{6.32}$$

and

$$A_XY = h'(Q'X, Q'Y) + h(Q'X, QY). \tag{6.33}$$

Now, we add (6.32) and (6.33) and obtain

$$T_XY + A_XY = h(X, QY) + h'(X, Q'Y). \tag{6.34}$$

Thus by using (6.15) and (6.34) we obtain

$$\widetilde{\nabla}_XY = \nabla^\circ_XY + T_XY + A_XY, \quad \forall\, X, Y \in \Gamma(TM), \tag{6.35}$$

which relates the Levi–Civita and Schouten–Van Kampen connections on (M, g) via the tensor fields T and A. Moreover, (6.16) becomes

$$\nabla^\circ_XY = \nabla^*_XY + A_{Q'Y}QX + T_{QY}Q'X. \tag{6.36}$$

Therefore all the relations between curvature tensor fields of the linear connections we defined on M, $\mathcal{D}$ and $\mathcal{D}^\perp$ can be expressed in terms of T and A. As an example we will transform (6.18) into such a formula. First from (6.35) we deduce that

$$\begin{aligned}\widetilde{\nabla}_X\widetilde{\nabla}_YZ &= \nabla^\circ_X\nabla^\circ_YZ + \nabla^\circ_X(T_YZ) + \nabla^\circ_X(A_YZ) + T_X(\nabla^\circ_YZ)\\ &+A_X(\nabla^\circ_YZ) + T_X(T_YZ) + A_X(T_YZ) + T_X(A_YZ) + A_X(A_YZ).\end{aligned} \tag{6.37}$$

Also, by using (6.35) and taking into account that ∇° has torsion tensor field T°, we obtain

$$\begin{aligned}\widetilde{\nabla}_{[X,Y]}Z &= \nabla^\circ_{[X,Y]}Z + T_{[X,Y]}Z + A_{[X,Y]}Z\\ &= \nabla^\circ_{[X,Y]}Z + T_{\nabla^\circ_XY}Z - T_{\nabla^\circ_YX}Z - T_{T^\circ(X,Y)}Z\\ &+ A_{\nabla^\circ_XY}Z - A_{\nabla^\circ_YX}Z - A_{T^\circ(X,Y)}Z.\end{aligned} \tag{6.38}$$

Next, by using ∇° we define covariant derivatives of T and A as follows:

$$(\nabla^\circ_XT)_YZ = \nabla^\circ_X(T_YZ) - T_{\nabla^\circ_XY}Z - T_Y(\nabla^\circ_XZ), \tag{6.39}$$

and

$$(\nabla^\circ_X A)_Y Z = \nabla^\circ_X (A_Y Z) - A_{\nabla^\circ_X Y} Z - A_Y(\nabla^\circ_X Z). \tag{6.40}$$

Then by direct calculations using (6.37)–(6.40) we obtain

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R^\circ(X,Y)Z + (\nabla^\circ_X T)_Y Z - (\nabla^\circ_Y T)_X Z + (\nabla^\circ_X A)_Y Z \\ &- (\nabla^\circ_Y A)_X Z + T_X(T_Y Z) - T_Y(T_X Z) + A_X(T_Y Z) - A_Y(T_X Z) \\ &+ T_X(A_Y Z) - T_Y(A_X Z) + A_X(A_Y Z) - A_Y(A_X Z) \\ &+ T_{T^\circ(X,Y)} Z + A_{T^\circ(X,Y)} Z, \ \forall X,Y,Z \in \Gamma(TM),\end{aligned} \tag{6.41}$$

which is not simpler than (6.18). However if we introduce a new tensor field B of type $(1,2)$ on M by

$$B(X,Y) = h(X,QY) + h'(X,Q'Y) = T_X Y + A_X Y, \tag{6.42}$$

for any $X, Y \in \Gamma(TM)$, then we can find a simpler relation than both (6.18) and (6.41). More precisely, by similar calculations we obtain the following formula

$$\begin{aligned}\widetilde{R}(X,Y)Z = R^\circ(X,Y)Z &+ (\nabla^\circ_X B)(Y,Z) - (\nabla^\circ_Y B)(X,Z) \\ &+ B(X, B(Y,Z)) - B(Y, B(X,Z)) + B(T^\circ(X,Y),Z),\end{aligned} \tag{6.43}$$

for any $X, Y, Z \in \Gamma(TM)$, where the covariant derivative of B is given by

$$(\nabla^\circ_X B)(Y,Z) = \nabla^\circ_X (B(Y,Z)) - B(\nabla^\circ_X Y, Z) - B(Y, \nabla^\circ_X Z). \tag{6.44}$$

From (6.42) and (6.44) we deduce that

$$\begin{aligned}(\nabla^\circ_X B)(Y,Z) &= (\nabla^\perp_X h)(Y,QZ) + (\nabla_X h')(Y,Q'Z) \\ &= (\nabla^\circ_X T)_Y Z + (\nabla^\circ_X A)_Y Z.\end{aligned} \tag{6.45}$$

Finally, we remark that (6.43) is useful when we work with the Schouten–Van Kampen and Levi–Civita connections. However, (6.18) is more efficient when the work concerns one of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$.

1.7 Sectional Curvatures of a Semi–Riemannian Non–Holonomic Manifold

Let $(M, g, \mathcal{D})$ be a semi–Riemannian non–holonomic manifold, that is, g is a semi–Riemannian metric on M and $\mathcal{D}$ is a non–integrable semi–Riemannian distribution on M. We show here that the restriction of the curvature tensor field R^* of the Vrănceanu connection $\nabla^* = (D, D^\perp)$ to $\Gamma(\mathcal{D})$ has the same properties as the curvature tensor $\widetilde{R}$ of Levi–Civita connection $\widetilde{\nabla}$ on (M, g), provided that g is parallel with respect to D. As a consequence we define the Vrănceanu sectional curvature of $(M, g, \mathcal{D})$ and prove that it determines R^*. Then R^* turns out to be as in (7.15) when $(M, g, \mathcal{D})$ is of constant Vrănceanu sectional curvature c. We also define the Schouten–Van Kampen sectional curvature of $(M, g, \mathcal{D})$ and study the relationship between these sectional curvatures and the sectional curvature of the ambient manifold. Finally, we find a large class of Riemannian non–holonomic manifolds of positive constant Vrănceanu sectional curvature.

Throughout this section we suppose that the semi–Riemannian metric g of $(M, g, \mathcal{D})$ is parallel with respect to the intrinsic connection D on $\mathcal{D}$. By the assertion (iii) of Theorem 5.9, this occurs if and only if the second fundamental form h of $\mathcal{D}$ satisfies

$$h(QX, QY) + h(QY, QX) = 0, \ \forall X, Y \in \Gamma(TM). \tag{7.1}$$

Since D is just the restriction of the Vrănceanu connection ∇^* to $\mathcal{D}$ we say, in this case, that g is **Vrănceanu–parallel** on $\mathcal{D}$. By Example 5.1 we see that $\mathcal{D}$ is not necessarily integrable.

Now, we consider the Vrănceanu connection ∇^* induced on M by the Levi–Civita connection $\widetilde{\nabla}$ on (M, g). Then we recall the Bianchi 1^{st} identity for ∇^* (cf. Kobayashi–Nomizu, [KN63], p. 135)

$$\sum_{(X,Y,Z)} \{(\nabla^*_X T^*)(Y, Z) + T^*(T^*(X, Y), Z) - R^*(X, Y)Z\} = 0, \tag{7.2}$$

for any $X, Y, Z \in \Gamma(TM)$, where T^* is the torsion of ∇^*. Also, by means of the curvature tensor field R^* of ∇^* we define the multilinear $F(M)$–mapping

$$\begin{aligned} R^* : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) &\longrightarrow F(M), \\ R^*(QU, QZ, QX, QY) &= g(R^*(QX, QY)QZ, QU), \end{aligned} \tag{7.3}$$

for any $X, Y, Z, U \in \Gamma(TM)$, and call it the **Vrănceanu curvature tensor field** of $(\mathcal{D}, g)$. Some of the most important properties of R^* are stated in the next lemma.

Lemma 7.1. *Let $(M, g, \mathcal{D})$ be a non–holonomic manifold such that g is Vrănceanu–parallel on $\mathcal{D}$. Then the Vrănceanu curvature tensor field of $\mathcal{D}$ satisfies:*

$$R^*(QU, QZ, QX, QY) + R^*(QU, QZ, QY, QX) = 0, \tag{7.4}$$

$$R^*(QU, QZ, QX, QY) + R^*(QZ, QU, QX, QY) = 0, \tag{7.5}$$

$$\sum_{(QZ,QX,QY)} \{R^*(QU, QZ, QX, QY)\} = 0, \tag{7.6}$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Proof. First, (7.4) is a well known property of the curvature tensor field of any linear connection on M. Next, by using (6.28) and (6.14) we deduce that

$$\begin{aligned} R^\circ(QX, QY)QZ &= R^*(QX, QY)QZ + h'(QZ, Q'T^*(QX, QY)) \\ &= R^*(QX, QY)QZ + h'(QZ, h(QY, QX) - h(QX, QY)). \end{aligned}$$

Then by (7.3) and (5.23) we obtain

$$\begin{aligned} & g(R^\circ(QX, QY)QZ, QU) = R^*(QU, QZ, QX, QY) \\ & + g(h(QZ, QU), h(QX, QY) - h(QY, QX)). \end{aligned} \tag{7.7}$$

Since g is parallel with respect to $\nabla^\circ = (\nabla, \nabla^\perp)$ (cf. (i) of Lemma 5.5) we have

$$g(R^\circ(QX, QY)QZ, QU) + g(R^\circ(QX, QY)QU, QZ) = 0. \tag{7.8}$$

Thus (7.7) and (7.8) imply

$$\begin{aligned} & R^*(QU, QZ, QX, QY) + R^*(QZ, QU, QX, QY) \\ & + g(h(QZ, QU) + h(QU, QZ), h(QX, QY) - h(QY, QX)) = 0. \end{aligned} \tag{7.9}$$

Then (7.5) follows from (7.9) via (7.1). Next, from (7.2) we infer that

$$\begin{aligned} \sum_{(QX, QY, QZ)} & \{(\nabla^*_{QX} T^*)(QY, QZ) + T^*(T^*(QX, QY), QZ) \\ & - R^*(QX, QY)QZ\} = 0. \end{aligned} \tag{7.10}$$

Taking into account (6.14) we obtain $T^*(T^*(QX, QY), QZ) = 0$, since $T^*(QX, QY) \in \Gamma(\mathcal{D}^\perp)$ and $T^*(Q'U, QV) = 0$ for any $U, V \in \Gamma(TM)$. Moreover, by using again (6.14) and taking into account that ∇^* is an adapted linear connection on $(M, \mathcal{D}, \mathcal{D}^\perp)$, we deduce that

$$(\nabla^*_{QX} T^*)(QY, QZ) \in \Gamma(\mathcal{D}^\perp).$$

On the other hand, we have

$$R^*(QX, QY)QZ \in \Gamma(\mathcal{D}),$$

since ∇^* is adapted to $\mathcal{D}$. Hence taking the $\mathcal{D}$ – and $\mathcal{D}^\perp$ – components in (7.10) we obtain

$$\begin{aligned} & \text{(a)} \quad \sum_{(QX, QY, QZ)} \{R^*(QX, QY)QZ\} = 0, \\ & \text{(b)} \quad \sum_{(QX, QY, QZ)} \{(\nabla^*_{QX} T^*)(QY, QZ)\} = 0. \end{aligned} \tag{7.11}$$

Then (7.6) follows from (7.11a) via (7.3). ∎

Corollary 7.2. *Let* $(M, g, \mathcal{D})$ *be as in Lemma* 7.1*. Then* R^* *satisfies*

$$R^*(QX, QY, QZ, QU) = R^*(QZ, QU, QX, QY), \tag{7.12}$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Proof. Denote the left hand side in (7.6) by $S^*(QU, QZ, QX, QY)$. Then by direct calculations using (7.4) and (7.5) we obtain

$$\begin{aligned} 0 &= S^*(QZ, QU, QX, QY) - S^*(QU, QX, QY, QZ) \\ &\quad - S^*(QX, QY, QZ, QU) + S^*(QY, QZ, QU, QX) \\ &= 2R^*(QZ, QU, QX, QY) - 2R^*(QX, QY, QZ, QU), \end{aligned}$$

which completes the proof of the corollary. ■

Next, we consider a 2–dimensional subspace W of $\mathcal{D}_x$ which we call a $\mathcal{D}$**–plane** at $x \in M$. For any basis $\{u, v\}$ of W we define

$$\Delta(u, v) = g(u, u)g(v, v) - g(u, v)^2. \tag{7.13}$$

As the matrix of the restriction of g to W with respect to the basis $\{u, v\}$ is

$$\begin{bmatrix} g(u,u) & g(u,v) \\ g(u,v) & g(v,v) \end{bmatrix},$$

we deduce that W is a non–degenerate subspace if and only if $\Delta(u, v) \neq 0$. For the basis $\{u, v\}$ of W we define the number

$$K^*(u \wedge v) = \frac{R^*(u, v, u, v)}{\Delta(u, v)},$$

provided W is non–degenerate.

If $\{u^*, v^*\}$ is another basis of W then $K^*(u \wedge v) = K^*(u^* \wedge v^*)$. Indeed, if we have

$$u^* = \alpha u + \beta v, \quad v^* = \gamma u + \delta v, \quad \alpha\delta - \beta\gamma \neq 0,$$

then, by using (7.4), (7.5) and (7.13) we deduce that

$$R^*(u^*, v^*, u^*, v^*) = (\alpha\delta - \beta\gamma)^2 R^*(u, v, u, v),$$

and

$$\Delta(u^*, v^*) = (\alpha\delta - \beta\gamma)^2 \Delta(u, v).$$

Thus $K^*(u \wedge v)$ is the same for any basis $\{u, v\}$ of W. This enables us to assign to any non–degenerate plane W of $\mathcal{D}_x$ the number

$$K^*(W) = \frac{R^*(u, v, u, v)}{\Delta(u, v)}, \tag{7.14}$$

where $\{u, v\}$ is an arbitrary basis of W. Then the **Vrănceanu sectional curvature** of the semi–Riemannian non–holonomic manifold $(M, g, \mathcal{D})$ is a real–valued function K^* on the set of all non–degenerate $\mathcal{D}$–planes given by (7.14). It is noteworthy that as in the case of semi–Riemannian manifolds, the Vrănceanu sectional curvature K^* of $(M, g, \mathcal{D})$ determines the curvature tensor field R^* of $\mathcal{D}$. To see this we first consider a 4–linear mapping $F : \mathcal{D}_x \times \mathcal{D}_x \times \mathcal{D}_x \times \mathcal{D}_x \longrightarrow \mathbb{R}$ that satisfies the four identities (7.4), (7.5), (7.6) and (7.12) which we proved for R^*. We call F a $\mathcal{D}$–**curvature–like mapping**. Then we state the following.

Lemma 7.3. *If $F(u, v, u, v) = 0$ for any $u, v \in \mathcal{D}_x$ spanning a non–degenerate $\mathcal{D}$–plane, then $F = 0$.*

The proof of this lemma is based on the four algebraic identities satisfied by F, and follows the same lines as the proof of Proposition 4.1 in O'Neill [O83], p. 78. For this reason we will omit it here. The following corollary is a straightforward consequence of Lemma 7.3.

Corollary 7.4. *Let F be a $\mathcal{D}$–curvature–like mapping such that*

$$K^*(u \wedge v) = \frac{F(u, v, u, v)}{\Delta(u, v)},$$

whenever $\{u, v\}$ spans a non–degenerate $\mathcal{D}$–plane. Then at any $x \in M$ we have

$$R^*(u, v, w, z) = F(u, v, w, z),$$

for any $u, v, w, z \in \mathcal{D}_x$.

If the Vrănceanu sectional curvature function K^* is a constant on M, then we say that the non–holonomic manifold $(M, g, \mathcal{D})$ is of **constant Vrănceanu curvature**. In this case the Vrănceanu curvature tensor field has a special form as it is stated in the next theorem.

Theorem 7.5. *Let $(M, g, \mathcal{D})$ be a semi–Riemannian non–holonomic manifold of constant Vrănceanu curvature c. Then the Vrănceanu curvature tensor field R^* of $\mathcal{D}$ has the form*

$$\begin{aligned} R^*(QX, QY, QZ, QU) = c\{&g(QX, QZ)g(QY, QU) \\ &- g(QX, QU)g(QY, QZ)\}, \end{aligned} \tag{7.15}$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Proof. Denote the right hand side in (7.15) by $F(QX, QY, QZ, QU)$. Then it is easy to check that F is a $\mathcal{D}$–curvature–like mapping. Moreover, we have

$$c = \frac{F(QX, QY, QX, QY)}{\Delta(QX, QY)},$$

for any $\{QX, QY\}$ that spans non–degenerate $\mathcal{D}$–planes. Thus (7.15) follows from Corollary 7.4. ■

In a similar way as the above theory was developed for the Vrănceanu connection ∇^* we may proceed with a theory for the Schouten–Van Kampen connection ∇°. We remark that in this case (7.6) and therefore (7.12) are not satisfied by R°. Thus we define the **Schouten–Van Kampen sectional curvature** K° of $(M, g, \mathcal{D})$ by a formula as in (7.14), that is,

$$K^\circ(W) = \frac{R^\circ(u, v, u, v)}{\Delta(u, v)}, \tag{7.16}$$

but we can not claim that K° determines R° on $\mathcal{D}$. This is another reason for saying that the Vrănceanu connection is more intimately related to the geometry of non–holonomic manifolds.

Now, we denote by $\widetilde{K}$ the sectional curvature of the semi–Riemannian manifold (M, g) defined by similar formulas as (7.14) or (7.16) but using the curvature tensor field $\widetilde{R}$ of the Levi–Civita connection on (M, g) (see O'Neill [O83], p. 77). Then we can relate $\widetilde{K}, K^*$ and K° as in the next theorem.

Theorem 7.6. *Let $(M, g, \mathcal{D})$ be a semi–Riemannian non–holonomic manifold such that g is Vrănceanu–parallel on $\mathcal{D}$. Then we have the following equalities:*

$$\widetilde{K}(W) = K^\circ(W) - \frac{g(h(QX, QY), h(QX, QY))}{\Delta(QX, QY)}, \tag{7.17}$$

$$K^\circ(W) = K^*(W) - 2\ \frac{g(h(QX, QY), h(QX, QY))}{\Delta(QX, QY)}, \tag{7.18}$$

$$\widetilde{K}(W) = K^*(W) - 3\ \frac{g(h(QX, QY), h(QX, QY))}{\Delta(QX, QY)}, \tag{7.19}$$

where $\{QX, QY\}$ is an arbitrary basis of the non–degenerate $\mathcal{D}$–plane W.

Proof. Replace (X, Y, QZ, QU) from (6.3) by $\{QX, QY, QY, QX\}$ and obtain

$$\widetilde{K}(W) = K^\circ(W) + \frac{g(h(QX, QY), h(QY, QX)) - g(h(QY, QY), h(QX, QX))}{\Delta(QX, QY)}.$$

Then taking into account (7.1) we obtain (7.17). Similarly, from (6.23) we deduce that

$$K^\circ(W) = K^*(W) + \frac{g(h'(QY, Q'T^*(QX, QY)), QX)}{\Delta(QX, QY)}. \tag{7.20}$$

Now, by using (5.23), (6.14) and (7.1) we infer that

$$\begin{aligned} g(h'(QY, Q'T^*(QX, QY)), QX) \\ &= -g(h(QY, QX), Q'T^*(QX, QY)) \\ &= -g(-h(QX, QY), -2h(QX, QY)) \\ &= -2g(h(QX, QY), h(QX, QY)). \end{aligned} \tag{7.21}$$

Thus by using (7.21) in (7.20) we obtain (7.18). Finally, (7.19) follows by using (7.18) into (7.17). ∎

Remark 7.1. As far as we know, O'Neill [O66] obtained first the equality (7.19) for the particular case of Riemannian submersions. The same equality was mentioned for foliations with bundle–like metrics (see (5.38c) in Tondeur [Ton97]. ∎

Now, suppose that $(M, g, \mathcal{D})$ is a Riemannian non–holonomic manifold. This means that $(\mathcal{D}, g)$ is a Riemannian distribution, but (M, g) might be proper semi–Riemannian manifold.

Corollary 7.7. *Let $(M, g, \mathcal{D})$ be a Riemannian non–holonomic manifold such that g is Vrănceanu–parallel on $\mathcal{D}$ and $h(QX, QY)$ is a space–like or light–like vector field for any two linearly independent vector fields $\{QX, QY\}$. Then we have*

$$\widetilde{K}(W) \leq K^{\circ}(W) \leq K^*(W), \tag{7.22}$$

for any non–degenerate $\mathcal{D}$–plane W.

Proof. In this case any W is a Euclidean subspace of $\mathcal{D}_x$ and therefore $\Delta(QX, QY) > 0$ for any $\{QX, QY\}$ spanning W. Also, by the hypothesis we have

$$g(h(QX, QY), h(QX, QY)) \geq 0.$$

Thus (7.22) follows from (7.17) and (7.18). ∎

Corollary 7.8. *Let M be an open submanifold of the Euclidean space $(\mathbb{R}^m, g)$ and $(M, g, \mathcal{D})$ be a Riemannian non–holonomic manifold such that g is Vrănceanu–parallel on $\mathcal{D}$. Then we have the assertions:*

(i) *At any point of M, both the Schouten–Van Kampen and Vrănceanu sectional curvatures must be non–negative.*
(ii) *If $(M, g, \mathcal{D})$ is a Riemannian non–holonomic manifold of constant Vrănceanu curvature c, then $c > 0$.*

Proof. As (M, g) is a Riemannian manifold, $h(QX, QY)$ is space–like, and thus the assertion (i) follows from (7.22) since $\widetilde{K}(W) = 0$. From (i) we deduce that $(M, g, \mathcal{D})$ can not be of negative constant Vrănceanu curvature. Finally, if $(M, g, \mathcal{D})$ is of zero Vrănceanu sectional curvature, then from (7.19) we deduce that h vanishes identically on M. By assertion (iii) of Theorem 5.6 we deduce that $\mathcal{D}$ is integrable, which is a contradiction because we supposed that $(M, g, \mathcal{D})$ is non–holonomic. ■

According to this corollary it is interesting to see if there exist Riemannian non–holonomic manifolds of positive constant Vrănceanu curvature. Surprisingly, there are plenty of such manifolds in dimension 3. To show this we consider the Euclidean space $(\mathbb{R}^3, g)$ and for any $\alpha \in \mathbb{R}$ and any non–zero $k \in \mathbb{R}$, define the family of 3–dimensional manifolds

$$M_{(\alpha,k)} = \left\{ (x^1, x^2, x^3) \in \mathbb{R}^3 : 0 < k(x^2 + x^3) + \alpha < \frac{\pi}{2} \right\} \cdot$$

Now, we fix the pair (α, k) and define on $M_{(\alpha,k)}$ the function

$$f(x^1, x^2, x^3) = \sqrt{2}\, \tan(k(x^2 + x^3) + \alpha), \tag{7.23}$$

and then, the linearly independent vector fields:

$$X = f\, \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1} \quad \text{and} \quad Y = f\, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^1} \cdot \tag{7.24}$$

Consider on $M_{(\alpha,k)}$ the restriction of the Euclidean metric g of $\mathbb{R}^3$ and denote by $\mathcal{D}$ the distribution spanned by $\{X, Y\}$. Then we prove the following.

Lemma 7.9. *For any fixed pair* (α, k)*we have the assertions:*

(i) $(M_{(\alpha,k)}, g, \mathcal{D})$ *is a Riemannian non–holonomic manifold.*
(ii) g *is Vrănceanu–parallel on* $\mathcal{D}$*.*

Proof. By direct calculations using (7.24) we deduce that

$$[X, Y] = kf(t)f'(t) \left(\frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^2} \right), \tag{7.25}$$

where $f(t) = \sqrt{2} \tan t$ and $t = k(x^2 + x^3) + \alpha$. The complementary orthogonal distribution $\mathcal{D}^\perp$ to $\mathcal{D}$ is spanned by

$$Z = f\, \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3} \cdot \tag{7.26}$$

Then we obtain

$$g([X, Y], Z) = -2kf(t)f'(t) \neq 0,$$

since $k \neq 0$, and on $M_{(\alpha,k)}$ we have $f(t) > 0$ and $f'(t) > 0$. Hence $[X,Y] \notin \Gamma(\mathcal{D})$, which proves the assertion (i). Next, since we have

$$g(X, Y) = -1, \tag{7.27}$$

the condition for g to be Vrănceanu–parallel on $\mathcal{D}$ (see the first equality in (5.37)) becomes

$$g([Z, X], Y) + g([Z, Y], X) = 0.$$

This is a consequence of

$$[Z, X] = [Z, Y] = -kf(t)f'(t)\ \frac{\partial}{\partial x^1},$$

and

$$g\left(\frac{\partial}{\partial x^1}, Y\right) + g\left(\frac{\partial}{\partial x^1}, X\right) = 0.$$

Thus the proof is complete. ■

Next, by using (5.3), (7.24), (7.25) and (7.27) we obtain

$$\nabla^*_X Y = \nabla^*_Y Y = \frac{kf'}{2 + f^2}(X + (1 + f^2)Y), \tag{7.28}$$

and

$$\nabla^*_Y X = \nabla^*_X X = \frac{kf'}{2 + f^2}\left((1 + f^2)X + Y\right). \tag{7.29}$$

Then by using (7.28), (7.29) and (7.24) we deduce that

$$\nabla^*_X \nabla^*_Y Y - \nabla^*_Y \nabla^*_X Y = 0. \tag{7.30}$$

Now, we decompose $[X, Y]$ given by (7.25) with respect to the non–holonomic frame field $\{X, Y, Z\}$ and obtain

$$[X, Y] = \frac{kff'}{2 + f^2}\ (-fX + fY - 2Z). \tag{7.31}$$

Then, taking into account (7.31) and (7.28) we infer that

$$\nabla^*_{[X,Y]} Y = -\frac{2kff'}{2 + f^2}\ \nabla^*_Z Y. \tag{7.32}$$

On the other hand, by using (3.16), (7.26) and (7.28) we get

$$\nabla^*_Z Y = Q[Z, Y] = -kff'Q\left(\frac{\partial}{\partial x^1}\right) = -\frac{kff'}{2 + f^2}\ (Y - X).$$

Hence (7.32) becomes

$$\nabla^*_{[X,Y]} Y = \frac{2k^2 f^2 (f')^2}{(2 + f^2)^2}\ (Y - X). \tag{7.33}$$

Then (7.3), (7.30) and (7.33) imply

$$R^*(X,Y,X,Y) = \frac{2k^2 f^2 (f')^2}{2+f^2}. \tag{7.34}$$

By direct calculations using (7.13) and (7.24) we deduce that

$$\Delta(X,Y) = f^2(2+f^2). \tag{7.35}$$

Then by using (7.34) and (7.35) in (7.14) we obtain

$$K^*(\mathcal{D}) = k^2. \tag{7.36}$$

Hence by (7.36) and Lemma 7.9 we may state the following important result.

Theorem 7.10. *$(M_{(\alpha,k)}, g, \mathcal{D})$ is a Riemannian non–holonomic manifold of positive constant Vrănceanu curvature.*

We remark that the result stated in Theorem 7.10 is specific to the geometry of non–holonomic manifolds. Indeed, if $\mathcal{D}$ would be involutive, then by (7.1) and the assertion (iii) of Lemma 5.5 we deduce that h vanishes identically on M. Thus any local leaf of $\mathcal{D}$ must be totally geodesic immersed in $\mathbb{R}^3$, and therefore $\mathcal{D}$ is of constant Vrănceanu curvature $c = 0$.

1.8 Degenerate Distributions of Codimension One

Let (M, g) be an $(n+1)$–dimensional proper semi–Riemannian manifold of index $0 < q < n+1$, and $\mathcal{D}$ be a distribution on M. Then the vector bundle $L_s^2(\mathcal{D}, \mathbb{R})$ (see Section 1.4) has a global section g^* induced by g in a natural way:

$$g^*(X,Y) = g(X,Y), \quad \forall\, X, Y \in \Gamma(\mathcal{D}). \tag{8.1}$$

When g^* is non–degenerate, the pair $(\mathcal{D}, g^*)$ is a semi–Riemannian distribution whose preliminary study was done in Sections 1.5, 1.6 and 1.7. We present in this section a method for studying $(\mathcal{D}, g^*)$ when g^* is degenerate. To avoid some cumbersome calculations and to abide by the size of our book, we restrict ourselves to n–distributions, but the technique we develop here can be extended for any degenerate distribution.

Thus from now on, in this section, $\mathcal{D}$ is an n–distribution (distribution of codimension one) on the $(n+1)$–dimensional proper semi–Riemannian manifold (M, g). Then we denote by the same symbol g the induced global section of $L_s^2(\mathcal{D}, \mathbb{R})$ given by (8.1). As $\dim \mathcal{D}_x = n$, by (4.3) we deduce that $\dim \mathcal{D}_x^{\perp} = 1$. Then the null subspace of $\mathcal{D}_x$ is (cf. (4.8))

$$\mathcal{N}_x(\mathcal{D}_x, g_x) = \mathcal{D}_x \cap \mathcal{D}_x^{\perp}, \tag{8.2}$$

and hence *null* $(\mathcal{D}_x, g_x) \leq 1$. According to (i) of Lemma 4.5, $\mathcal{D}_x$ is degenerate if and only if $null(\mathcal{D}_x, g_x) > 0$, which in our case is equivalent to $null(\mathcal{D}_x, g_x) = 1$. When $\mathcal{D}_x$ is a degenerate subspace of T_xM for all $x \in M$, we say that $(\mathcal{D}, g)$ is a **degenerate distribution** on (M, g). Now, consider the null distribution $\mathcal{N}(\mathcal{D}, g)$ and the orthogonal distribution $\mathcal{D}^\perp$, that is,

$$\mathcal{N}(\mathcal{D}, g) = \bigcup_{x \in M} \mathcal{N}_x(\mathcal{D}_x, g_x), \quad \mathcal{D}^\perp = \bigcup_{x \in M} \mathcal{D}_x^\perp.$$

Then based on the above discussion we can characterize degenerate distributions in terms of null and orthogonal distributions as follows.

Theorem 8.1. *Let (M, g) be an $(n+1)$–dimensional proper semi–Riemannian manifold and $\mathcal{D}$ be an n–distribution on M. Then the following assertions are equivalent:*

(i) *$(\mathcal{D}, g)$ is a degenerate distribution.*
(ii) *The null distribution of $\mathcal{D}$ coincides with the orthogonal distribution to $\mathcal{D}$.*
(iii) *The orthogonal distribution to $\mathcal{D}$ is a vector subbundle of $\mathcal{D}$.*

Thus by (ii) we have

$$\Gamma(\mathcal{D}^\perp) = \{\xi \in \Gamma(\mathcal{D}) : g(\xi, X) = 0, \quad \forall\, X \in \Gamma(\mathcal{D})\}, \tag{8.3}$$

which entitles us to call $\mathcal{D}^\perp$ the **null distribution** of $(\mathcal{D}, g)$.

Next, we consider $n = 1$ and note that $\mathcal{D}_x$ is a totally null subspace of T_xM for all $x \in M$. Then $\mathcal{D}^\perp = \mathcal{D}$ and therefore $\mathcal{D}$ is a totally–null distribution on (M, g). As M is now a 2–dimensional proper semi–Riemannian manifold we conclude that (M, g) must be a Lorentz manifold. Since $\mathcal{D}^\perp$ is not anymore complementary to $\mathcal{D}$ in TM, to proceed with the study of $(\mathcal{D}, g)$ we need a transversal vector bundle to $\mathcal{D}$ in TM which, of course, can not be orthogonal to $\mathcal{D}$. To achieve this, we consider a complementary distribution $\mathcal{H}$ to $\mathcal{D}$ in TM. Then on a coordinate neighbourhood $\mathcal{U} \subset M$ take the local sections $\xi \in \Gamma(\mathcal{D}_{|\mathcal{U}})$ and $Z \in \Gamma(\mathcal{H}_{|\mathcal{U}})$. Note that $g(\xi, Z) \neq 0$ on $\mathcal{U}$, otherwise g is not a Lorentz metric on M. Now, we define on $\mathcal{U}$ the vector field

$$V = \frac{1}{g(\xi, Z)} \left\{ Z - \frac{g(Z, Z)}{2g(\xi, Z)}\, \xi \right\}. \tag{8.4}$$

It is easy to check that V satisfies

$$\text{(a)} \;\; g(V, V) = 0 \;\; \text{and} \;\; \text{(b)} \;\; g(V, \xi) = 1. \tag{8.5}$$

If $\mathcal{U}^*$ is another coordinate neighbourhood on M such that $\mathcal{U} \cap \mathcal{U}^* \neq \emptyset$, then by direct calculations using (8.4) for both neighbourhoods, we deduce that $V^* = fV$, where f is a smooth function on $\mathcal{U} \cap \mathcal{U}^*$. Thus there exists a distribution $\mathcal{D}'$ on M which is locally spanned by the vector field given by

(8.4). For any other complementary distribution to $\mathcal{D}$ in TM, (8.4) defines the same distribution $\mathcal{D}'$ on M. Also, from (8.5) we deduce that $\mathcal{D}'$ is a totally null distribution that is complementary to $\mathcal{D}$ in TM. On the other hand, it is easy to check that any vector field V on $\mathcal{U}$ satisfying (8.5) must be given by (8.4). Thus $\mathcal{D}'$ is the only totally null complementary distribution to $\mathcal{D}$ in TM. This discussion enables us to state the following

Theorem 8.2. *Let (M, g) be a 2–dimensional Lorentz manifold and $\mathcal{D}$ be a totally–null distribution on M. Then there exists a unique totally–null distribution $\mathcal{D}'$ that is complementary to $\mathcal{D}$ in TM.*

We call $\mathcal{D}'$ the **totally–null transversal distribution** to $\mathcal{D}$. Also, we call the pair $\{\xi, V\}$ the **null frame** with respect to the decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}'. \tag{8.6}$$

Now, we consider the Levi–Civita connection $\widetilde{\nabla}$ on (M, g). Then, with respect to the null frame $\{\xi, V\}$ we put

$$\text{(a)} \quad \widetilde{\nabla}_\xi \xi = \alpha\xi + \beta V \quad \text{and} \quad \text{(b)} \quad \widetilde{\nabla}_\xi V = \gamma\xi + \delta V, \tag{8.7}$$

where $\alpha, \beta, \gamma, \delta$ are smooth functions on a coordinate neighbourhood $\mathcal{U} \subset M$. By using (8.7a) and (8.5b) we obtain

$$\beta = g(\widetilde{\nabla}_\xi \xi, \xi) - \alpha g(\xi, \xi) = 0,$$

since ξ is a light–like vector field and g is parallel with respect to $\widetilde{\nabla}$. Similarly we deduce that $\gamma = 0$. Moreover, by (8.5) and (5.9) we have

$$\alpha = g(\widetilde{\nabla}_\xi \xi, V) = -g(\xi, \widetilde{\nabla}_\xi V) = -\delta.$$

Thus (8.7) becomes

$$\text{(a)} \quad \widetilde{\nabla}_\xi \xi = a\xi \quad \text{and} \quad \text{(b)} \quad \widetilde{\nabla}_\xi V = -aV, \tag{8.8}$$

where a is a smooth function on $\mathcal{U}$. Next, we consider an integral curve $C : x^a = x^a(t)$ of ξ and from (8.8a) we deduce the differential equations

$$\frac{d^2x^a}{dt^2} + \left\{ {}_b{}^a{}_c \right\} (x(t)) \, \frac{dx^b}{dt} \, \frac{dx^c}{dt} = a(x(t)) \, \frac{dx^a}{dt}, \tag{8.9}$$

where $\left\{ {}_b{}^a{}_c \right\}$ are the Christoffel coefficients for $\widetilde{\nabla}$, and $a, b, c \in \{1, 2\}$. Then we take a new parameter s on C satisfying the differential equation

$$\frac{d^2s}{dt^2} - a(x(t)) \, \frac{ds}{dt} = 0.$$

The existence of s is guaranteed by the general theorem of existence and uniqueness for differential equations (cf. Theorem 2.1.2). Thus (8.9) becomes

$$\frac{d^2x^a}{ds^2} + \left\{ \begin{matrix} a \\ b \quad c \end{matrix} \right\} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0,$$

that is, with respect to this parametrization, C is a null geodesic of (M, g). According to O'Neill [O83], p. 69, a curve in (M, g) that becomes a geodesic after a reparametrization, is called a **pregeodesic**. Hence any integral curve of $\mathcal{D}$ is a null pregeodesic.

It is easily seen that the above discussion about $\mathcal{D}$ is also valid for $\mathcal{D}'$. In particular, it follows that

$$\text{(a)} \quad \widetilde{\nabla}_V V = bV \quad \text{and} \quad \text{(b)} \quad \widetilde{\nabla}_V \xi = -b\xi. \tag{8.10}$$

Then by (8.8) and (8.10) we deduce that $\widetilde{\nabla}$ is an adapted connection on the almost product manifold $(M, \mathcal{D}, \mathcal{D}')$. Thus in case $n = 1$, the whole geometry of $(\mathcal{D}, g)$ can be summarized in the following theorem.

Theorem 8.3. *Let (M, g) be a 2–dimensional Lorentz manifold and $\mathcal{D}$ be a totally–null line field on M. Then we have the following assertions:*

(i) *The Levi–Civita connection $\widetilde{\nabla}$ on (M, g) is an adapted connection on the almost product manifold $(M, \mathcal{D}, \mathcal{D}')$.*
(ii) *The integral curves of both $\mathcal{D}$ and $\mathcal{D}'$ are null pregeodesics of (M, g).*

Next, we consider the case $n > 1$. By (iii) of Theorem 8.1, $\mathcal{D}^\perp$ is a vector subbundle of $\mathcal{D}$ and therefore any complementary distribution to $\mathcal{D}$ in TM is not orthogonal to $\mathcal{D}$. Moreover $\mathcal{D}$ is a partially–null distribution on M because it is a 1-degenerate n–distribution with $n > 1$. As in the case $n = 1$ we look for a totally–null transversal distribution to $\mathcal{D}$. To this end, we consider a complementary distribution $\mathcal{S}$ to $\mathcal{D}^\perp$ in $\mathcal{D}$, that is, we have

$$\mathcal{D} = \mathcal{S} \oplus \mathcal{D}^\perp. \tag{8.11}$$

As any $\mathcal{S}_x$ is a screen subspace of $\mathcal{D}_x$, we call $\mathcal{S}$ a **screen distribution** for $\mathcal{D}$. By Lemma 4.6 we deduce that $\mathcal{S}$ is a non–degenerate distribution on M. Then by Lemma 4.3 we deduce that the complementary orthogonal distribution $\mathcal{S}^\perp$ to $\mathcal{S}$ in TM is non–degenerate too. Therefore, $\mathcal{S}^\perp$ is a 2–distribution satisfying

$$TM = \mathcal{S} \oplus \mathcal{S}^\perp, \tag{8.12}$$

and $\mathcal{D}^\perp$ is a vector subbundle of $\mathcal{S}^\perp$. Then we consider a complementary distribution $\mathcal{H}$ to $\mathcal{D}^\perp$ in $\mathcal{S}^\perp$. Take the nowhere zero local sections $\xi \in \Gamma(\mathcal{D}^\perp_{|\mathcal{U}})$ and $Z \in \Gamma(\mathcal{H}_{|\mathcal{U}})$ and observe that $g(\xi, Z) \neq 0$ on $\mathcal{U}$ since $\mathcal{S}^\perp$ is non–degenerate. Then define the vector field V by (8.4) and following the same steps as in the study we developed for $n = 1$, we obtain a totally null 1–distribution $\mathcal{D}'(\mathcal{S})$ that is complementary to $\mathcal{D}$ in TM. Hence by (8.11) and (8.12) we have

$$TM = \mathcal{D} \oplus \mathcal{D}'(\mathcal{S}) = \mathcal{S} \oplus \mathcal{D}^\perp \oplus \mathcal{D}'(\mathcal{S}). \tag{8.13}$$

It follows that $\mathcal{D}'(\mathcal{S})$ is the only distribution on M whose local section V satisfies (8.5) and

$$g(V, X) = 0, \quad \forall\, X \in \Gamma(\mathcal{S}). \tag{8.14}$$

The above study can be summarized as follows.

Theorem 8.4. *Let $\mathcal{D}$ be a partially–null n–distribution on an $(n+1)$–dimensional proper semi–Riemannian manifold (M, g) with $n > 1$. Then for a screen distribution $\mathcal{S}$ on M there exists a unique totally–null 1–distribution $\mathcal{D}'(\mathcal{S})$ that is complementary to $\mathcal{D}$ in TM.*

We call $\mathcal{D}'(\mathcal{S})$ the **totally–null transversal distribution** to $\mathcal{D}$ with respect to the screen distribution $\mathcal{S}$.

Example 8.1. Consider the Lorentz space $(\mathbb{R}^2_1, g)$ whose metric is given by (4.10) for $m = 2$. Then with respect to the rectangular coordinates (x^1, x^2) on $\mathbb{R}^2$, any degenerate distribution $\mathcal{D}$ on $\mathbb{R}^2$ is spanned by

$$\xi = \frac{\partial}{\partial x^1} + \varepsilon\, \frac{\partial}{\partial x^2}, \quad \varepsilon = \pm 1.$$

By (8.4) we deduce that the totally–null transversal distribution $\mathcal{D}'$ is spanned by

$$V = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - \varepsilon\, \frac{\partial}{\partial x^2}\right).$$

Clearly, the integral curves of both ξ and V are null geodesics of $(\mathbb{R}^2_1, g)$. ■

Example 8.2. On the Lorentz space $(\mathbb{R}^4_1, g)$ consider the 3–distribution $\mathcal{D}$ spanned by

$$\Big\{ X_1 = -\sqrt{2(1+e^{2x^1})}\, \frac{\partial}{\partial x^1} + \sqrt{1+e^{2x^1}}\, \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} - e^{x^1}\, \frac{\partial}{\partial x^4},$$
$$X_2 = \frac{\partial}{\partial x^1} - \sqrt{2}\, \frac{\partial}{\partial x^2}, \quad X_3 = e^{x^1}\, \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4} \Big\}.$$

It is easy to check that $\mathcal{D}$ is non–integrable and 1–degenerate with respect to the Lorentz metric (4.10) for $m = 4$. As $\xi = X_1$, we may take the screen distribution

$$\mathcal{S} = \mathrm{span}\{X_2, X_3\}.$$

Then by using (8.4) where Z is replaced either by X_2 or by X_3, we deduce that the totally–null transversal distribution $\mathcal{D}'(\mathcal{S})$ is spanned by

$$V = \frac{1}{2(1+e^{2x^1})}\left(\sqrt{2(1+e^{2x^1})}\, \frac{\partial}{\partial x^1} - \sqrt{1+e^{2x^1}}\, \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} - e^{x^1}\, \frac{\partial}{\partial x^4}\right). \quad ■$$

Remark 8.3. It is interesting to note that in Example 8.2 we may choose an integrable screen distribution

$$\mathcal{S}^* = \text{span}\left\{X_2^* = -\sqrt{1+e^{2x^1}}\,\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} - e^{x^1}\frac{\partial}{\partial x^4},\ X_3^* = e^{x^1}\frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4}\right\}.$$

However, at this moment, we do not have an answer to the question: can we find an integrable screen distribution for any partially–null distribution on a semi–Euclidean space $\mathbb{R}^m_q$? ∎

Now, we come back to the general theory on the geometry of a degenerate n–distribution $\mathcal{D}$ on the $(n+1)$–dimensional proper semi–Riemannian manifold (M,g), with $n>1$. Throughout the study, we consider the local non–holonomic null frame field $\{\xi, V\}$ on the distribution $D^{\perp}\oplus\mathcal{D}'(\mathcal{S})$, where V is given by (8.4) with Z from $\Gamma(\mathcal{S}^{\perp})$. Then we define the differential 1–forms ω and τ by

$$\begin{aligned}&\text{(a)}\quad \omega(X) = g(X,\xi) \quad \text{and}\\ &\text{(b)}\quad \tau(X) = g(X,V), \quad \forall\, X\in\Gamma(TM),\end{aligned} \tag{8.15}$$

and denote by Q and $\bar{Q}$ the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{S}$ with respect to the decompositions in (8.13). Thus by using (8.13) and (8.15) we write

$$\begin{aligned}&\text{(a)}\quad X = QX + \omega(X)V \quad \text{and}\\ &\text{(b)}\quad X = \bar{Q}X + \tau(X)\xi + \omega(X)V,\end{aligned} \tag{8.16}$$

for any $X\in\Gamma(TM)$.

Next, we consider the Levi–Civita connection $\widetilde{\nabla}$ on (M,g) and according to the first decomposition in (8.13) we put

$$\widetilde{\nabla}_X QY = \nabla_X QY + B(X,QY)V, \tag{8.17}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \eta(X)V, \quad \forall\, X,Y\in\Gamma(TM), \tag{8.18}$$

where $\nabla_X QY$ and $A_V X$ lie in $\Gamma(\mathcal{D})$. Then we have the following induced geometric objects:

$$\begin{aligned}
&\nabla_X : \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}) && \text{, a linear connection on } \mathcal{D},\\
&B \;\;: \Gamma(TM)\times\Gamma(\mathcal{D}) \longrightarrow F(M) && \text{, an } F(M)\text{–bilinear mapping},\\
&A_V : \Gamma(TM) \longrightarrow \Gamma(\mathcal{D}) && \text{, an } F(M)\text{–linear operator},\\
&\eta \;\;: \Gamma(TM) \longrightarrow F(M) && \text{, a 1–form on } M.
\end{aligned}$$

As in the case of semi–Riemannian distributions, we call ∇ the **induced connection** on $\mathcal{D}$ and

$$B : \Gamma(\mathcal{D})\times\Gamma(\mathcal{D}) \longrightarrow F(M); \;\; B(X,Y) = \omega(\widetilde{\nabla}_X Y), \;\; \forall\, X, Y \in \Gamma(\mathcal{D}), \tag{8.19}$$

the **second fundamental form** of $\mathcal{D}$. Also, A_V is called the **Weingarten operator** with respect to $V \in \Gamma(\mathcal{D}'(\mathcal{S}))$. By (8.15) we see that ω does not depend on the screen distribution. This implies the following important property of B.

Proposition 8.5. *The second fundamental form of the partially–null distribution $\mathcal{D}$ does not depend on the screen distribution.*

Taking into account the decomposition (8.11) we set

$$\nabla_X \bar{Q}Y = \overline{\nabla}_X \bar{Q}Y + \bar{B}(X, \bar{Q}Y)\xi, \;\; \forall\, X, Y \in \Gamma(TM), \tag{8.20}$$

where $\overline{\nabla}$ is a linear connection on the screen distribution given by

$$\overline{\nabla}_X \bar{Q}Y = \bar{Q}(\nabla_X \bar{Q}Y) = \bar{Q}(Q(\widetilde{\nabla}_X \bar{Q}Y)), \tag{8.21}$$

and $\bar{B}$ is an $F(M)$–bilinear mapping on $\Gamma(TM)\times\Gamma(\mathcal{S})$. We call $\overline{\nabla}$ the **induced connection** on the screen distribution $\mathcal{S}$ and

$$\bar{B} : \Gamma(\mathcal{S})\times\Gamma(\mathcal{S}) \longrightarrow F(M); \;\; \bar{B}(\bar{Q}X, \bar{Q}Y) = \tau(\nabla_{\bar{Q}X} \bar{Q}Y), \tag{8.22}$$

the **second fundamental form** of $\mathcal{S}$ in $\mathcal{D}$. Also, we put

$$\nabla_X \xi = -\bar{A}_\xi X + \overline{\eta}(X)\xi, \;\; \forall\, X \in \Gamma(TM), \tag{8.23}$$

where $\overline{\eta}$ is a 1–form on M and

$$\bar{A}_\xi : \Gamma(TM) \longrightarrow \Gamma(\mathcal{S}); \;\; \bar{A}_\xi X = \bar{Q}(\nabla_X \xi), \tag{8.24}$$

is an $F(M)$–linear operator. We call $\bar{A}_\xi$ the **Weingarten operator** of the screen distribution $\mathcal{S}$ with respect to ξ. In the next theorem we bring together the main properties of the geometric objects involved in the study of $\mathcal{D}$.

Theorem 8.6. *Let $\mathcal{D}$ be a partially–null n–distribution on an* $(n+1)$*–dimensional proper semi–Riemannian manifold* (M, g) *with* $n > 1$*, and $\mathcal{S}$ be a screen distribution of $\mathcal{D}$. If $B, \overline{B}, \eta, \overline{\eta}, A, \bar{A}, \nabla, \overline{\nabla}$ are as given by equations* (8.17)–(8.24)*, then we have:*

$$B(X, \xi) = 0, \tag{8.25}$$

$$\begin{aligned} &\text{(a)} \;\; \widetilde{\nabla}_X \xi = \nabla_X \xi, \\ &\text{(b)} \;\; \overline{\eta}(X) = -\eta(X), \end{aligned} \tag{8.26}$$

$$\begin{aligned} &\text{(a)} \quad B(X, QY) = g(\bar{A}_\xi X, QY), \\ &\text{(b)} \quad \bar{B}(X, \bar{Q}Y) = g(A_V X, \bar{Q}Y), \end{aligned} \tag{8.27}$$

$$(\nabla_X g)(QY, QZ) = B(X, QY)\tau(QZ) + B(X, QZ)\tau(QY), \tag{8.28}$$

$$(\overline{\nabla}_X g)(\bar{Q}Y, \bar{Q}Z) = 0, \tag{8.29}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. From (8.19) and (8.15a) we deduce that

$$B(X, \xi) = g(\widetilde{\nabla}_X \xi, \xi) = 0,$$

which proves (8.25). Then (8.26a) follows from (8.17) via (8.25). By using (8.23), (8.26a), (5.9) and (8.18) we obtain

$$\overline{\eta}(X) = g(\nabla_X \xi, V) = g(\widetilde{\nabla}_X \xi, V) = -g(\xi, \widetilde{\nabla}_X V) = -\eta(X),$$

which proves (8.26b). Next, we use (8.17), (8.5b), (8.26a) and (8.23) to obtain

$$B(X, QY) = g(\widetilde{\nabla}_X QY, \xi) = -g(QY, \nabla_X \xi) = g(\bar{A}_\xi X, QY),$$

which proves (8.27a). Similarly, by using (8.20), (8.5b), (8.17) and (8.18) we infer that

$$\bar{B}(X, \bar{Q}Y) = g(\nabla_X \bar{Q}Y, V) = -g(\bar{Q}Y, \widetilde{\nabla}_X V) = g(A_V X, \bar{Q}Y),$$

which is (8.27b). Finally, both (8.28) and (8.29) are consequences of (5.9) via (8.17), (8.15b) and (8.20). ∎

Remark 8.4. From (8.29) and (8.28) we see that g is parallel with respect to the induced connection on the screen distribution, but in general, it is not parallel with respect to the induced connection on $\mathcal{D}$. This makes the study of degenerate distributions very different and more difficult than that of non–degenerate distributions (see (i) of Lemma 5.5). ∎

According to this remark it seems more appropriate to use $\overline{\nabla}$ instead of ∇ in studying the geometry of $\mathcal{D}$. Thus from (8.17) and (8.20) we deduce that

$$\widetilde{\nabla}_X \bar{Q}Y = \overline{\nabla}_X \bar{Q}Y + \bar{B}(X, \bar{Q}Y)\xi + B(X, \bar{Q}Y)V. \tag{8.30}$$

Also, by using (8.26) and (8.23) we obtain

$$\widetilde{\nabla}_X \xi = -\bar{A}_\xi X - \eta(X)\xi, \tag{8.31}$$

where η is given by

$$\eta(X) = g(\widetilde{\nabla}_X V, \xi) = -g(V, \widetilde{\nabla}_X \xi). \tag{8.32}$$

Now, we can state the following.

Theorem 8.7. *Let $\mathcal{D}$ be as in Theorem* 8.6*. Then the following assertions are equivalent:*

(i) *The Levi–Civita connection $\widetilde{\nabla}$ on (M, g) is an adapted connection to $\mathcal{D}$.*
(ii) *For any $X, Y \in \Gamma(TM)$ we have*

$$B(X, \bar{Q}Y) = 0. \tag{8.33}$$

(iii) *$\bar{A}_\xi$ vanishes identically on M.*
(iv) *$\widetilde{\nabla}$ is an adapted connection to the null distribution $\mathcal{D}^\perp$.*

Proof. The equivalence of (i) and (ii) follows from (8.30) and (8.31). Then (ii) $\Longleftrightarrow$ (iii) is a consequence of (8.27a). Finally, since ξ is spanning the null distribution of $\mathcal{D}$, from (8.31) we deduce the equivalence of (iii) and (iv). ■

We now show an interesting relationship between the two geometries of degenerate and non–degenerate distributions.

Theorem 8.8. *Let $\mathcal{D}$ be as in Theorem* 8.6*. Then the following assertions are equivalent:*

(i) *The Levi–Civita connection $\widetilde{\nabla}$ on (M, g) is adapted to the screen distribution $\mathcal{S}$.*
(ii) *$\widetilde{\nabla}$ is an adapted linear connection on the almost product manifold $(M, \mathcal{D}, \mathcal{D}'(\mathcal{S}))$.*

Proof. From (8.30) we see that $\widetilde{\nabla}$ is adapted to $\mathcal{S}$ if and only if (8.33) is satisfied and

$$\bar{B}(X, \bar{Q}Y) = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{8.34}$$

From Theorem 8.7 we know that (8.33) holds if and only if $\widetilde{\nabla}$ is adapted to $\mathcal{D}$. Thus to complete the proof it is sufficient to prove that (8.34) holds if and only if $\widetilde{\nabla}$ is adapted to $\mathcal{D}'(\mathcal{S})$. First, suppose (8.34) is satisfied. Then from (8.27b) we see that $A_V X$ has no component in $\Gamma(\mathcal{S})$ for any $X \in \Gamma(TM)$. On the other hand, from (8.18) we deduce that

$$g(A_V X, V) = 0,$$

so $A_V X$ has no component in $\Gamma(\mathcal{D}^\perp)$. As $A_V X \in \Gamma(\mathcal{D})$ we conclude that $A_V X = 0$ for any $X \in \Gamma(TM)$. Then from (8.18) it follows that $\widetilde{\nabla}$ is adapted to $\mathcal{D}'(\mathcal{S})$. Conversely, if $\widetilde{\nabla}$ is adapted to $\mathcal{D}'(\mathcal{S})$, then from (8.18) we obtain (8.34) via (8.27b). This completes the proof of the theorem. ■

If $\widetilde{\nabla}$ is adapted to $\mathcal{D}$ then (8.17) implies $B = 0$ on M. Thus we have

$$[QX, QY] = \widetilde{\nabla}_{QX}QY - \widetilde{\nabla}_{QY}QX = \nabla_{QX}QY - \nabla_{QY}QX \in \Gamma(\mathcal{D}),$$

that is, $\mathcal{D}$ is integrable. As $\mathcal{D}$ is a degenerate distribution, all of its leaves are degenerate hypersurfaces of (M, g). Moreover, from (8.17) we deduce that any leaf of $\mathcal{D}$ is a totally geodesic degenerate hypersurface of (M, g). Also from (8.18) we see that every integral curve of $\mathcal{D}'(\mathcal{S})$ is a pregeodesic, provided $\widetilde{\nabla}$ is adapted to $\mathcal{D}'(\mathcal{S})$. Based on this discussion and by using Theorems 5.12 and 8.8 we state the following.

Corollary 8.9. *Suppose $\mathcal{D}$ has a screen distribution $\mathcal{S}$ and $\widetilde{\nabla}$ is adapted to $\mathcal{S}$. Then the semi–Riemannian manifold (M, g) is locally represented in the following two equivalent ways:*

(i) *It is a locally semi–Riemannian product of leaves of $\mathcal{S}$ and $\mathcal{S}^{\perp}$.*
(ii) *It is a locally product of a totally geodesic degenerate hypersurface with a pregeodesic of $\mathcal{D}'(\mathcal{S})$.*

2

STRUCTURAL AND TRANSVERSAL GEOMETRY OF FOLIATIONS

In this chapter we introduce the theory of foliations. The basic material on this theory is given in Section 2.1. Here all different classical approaches to foliations are given, that is to say we talk about foliations using foliated atlases, involutive distributions and differential forms. Then a list of examples is given showing that foliations appear in a natural way in the theory of submersions, non–singular systems of differential equations, fiber bundles, Lie group actions and CR–submanifolds of Kähler manifolds. Foliations will appear later in the book in many other areas like Finsler geometry, symplectic geometry, contact geometry, etc.

The second section sets the stage for studying the geometry of foliations by introducing adapted tensor fields on foliated manifolds, discussing their existence problem and determining their properties. Then we introduce in Section 2.3 the structural and transversal covariant derivatives induced by an adapted connection on a foliated manifold. The local components of both the torsion and curvature tensor fields with respect to a semi–holonomic frame field determine adapted tensor fields which are going to play an important role in studying the geometry of foliations. Finally, in the last section, by using both the structural and transversal covariant derivatives, we derive all Ricci and Bianchi identities for an adapted linear connection.

2.1 Definitions and Examples

Let $\mathbb{R}^m$ be the m–dimensional Euclidean space with the usual scalar product given by (1.4.11).Denote by τ the standard topology induced on $\mathbb{R}^m$ by the Euclidean norm

$$\|x\| = \sqrt{(x^1)^2 + \cdots + (x^m)^2}.$$

Thus open balls by this norm are neighbourhoods with respect to τ. Then $(\mathbb{R}^m, \tau)$ becomes an m–dimensional smooth manifold with a global chart $(\mathbb{R}^m, 1_{\mathbb{R}^m})$.

Next, we consider two positive integers n and p such that $m = n+p$. Then the space $\mathbb{R}^m$ can be identified with the Cartesian product $\mathbb{R}^n \times \mathbb{R}^p$ of the two spaces $\mathbb{R}^n$ and $\mathbb{R}^p$. If $c = (c^{n+1}, ..., c^{n+p})$ is a point of $\mathbb{R}^p$ we denote by $\mathbb{R}^n_c$ the affine n–dimensional subspace of $\mathbb{R}^m$ passing through the point $(0, ..., 0, c^{n+1}, ..., c^{n+p}) \in \mathbb{R}^m$, and parallel to $\mathbb{R}^n$, that is,

$$\mathbb{R}^n_c = \left\{(x^1, ..., x^m) \in \mathbb{R}^m : \ x^{n+1} = c^{n+1}, ..., x^{n+p} = c^{n+p}\right\}.$$

Then an (n, c)–**plaque** P^n_c in $\mathbb{R}^m$ is the intersection of $\mathbb{R}^n_c$ with an open ball in $\mathbb{R}^m$ with respect to the topology τ. For any n we define on $\mathbb{R}^m$ a new topology τ_n whose open basis consists of all (n, c)–plaques of $\mathbb{R}^m$. It is easy to see that $(\mathbb{R}^m, \tau_n)$ is a Hausdorff space and each $\mathbb{R}^n_c$ is both closed and open (clopen) in $\mathbb{R}^m$ with respect to τ_n. Finally, we remark that τ_n is just the product of the standard topology on $\mathbb{R}^n$ and the discrete topology on $\mathbb{R}^p$.

Now, we consider the following disjoint partition of $\mathbb{R}^m$:

$$\mathbb{R}^m = \bigcup_{c \in \mathbb{R}^p} \mathbb{R}^n_c. \tag{1.1}$$

This suggests the following definition. We say that the family $\{\mathbb{R}^n_c\}$, $c \in \mathbb{R}^p$ is a **foliation of dimension** n (or **codimension** p) of $\mathbb{R}^m$, and each $\mathbb{R}^n_c$ is a **leaf** of the foliation. It is worth mentioning that the leaves of the foliation are the connected components of $(\mathbb{R}^m, \tau_n)$, and that each such leaf is an n–dimensional submanifold of $(\mathbb{R}^m, \tau)$.

We also note that $(\mathbb{R}^m, \tau_n)$ becomes an n–dimensional smooth manifold. Indeed, we define on $\mathbb{R}^m$ a smooth atlas with local charts (P^n_c, Π^n_c), where

$$\Pi^n_c : P^n_c \longrightarrow \mathbb{R}^n; \ \Pi^n_c(x^1, ..., x^n, c^{n+1}, ..., c^{n+p}) = (x^1, ..., x^n).$$

The partition (1.1) of $\mathbb{R}^m$ can be generalized to smooth manifolds as follows. Let M be an m–dimensional manifold and $\mathcal{F} = \{L_t\}_{t \in I}$ be a family of connected subsets of M. Suppose that $\mathcal{F}$ is a **partition** of M, that is, we have

$$M = \bigcup_{t \in I} L_t \quad \text{and} \quad L_t \cap L_s = \emptyset, \quad \text{for } t \neq s. \tag{1.2}$$

Next, we consider a positive integer $n < m$ and a local chart $(\mathcal{U}, \varphi)$ on M. Then we say that $(\mathcal{U}, \varphi)$ is an n–**foliated chart**, if whenever $L_t \cap \mathcal{U} \neq \emptyset$ for some $t \in I$, then each connected component of $L_t \cap \mathcal{U}$ is mapped by φ onto an (n, c)–plaque of $\mathbb{R}^m$. An n–**foliated atlas** associated to $\mathcal{F}$ on M is a collection of n–foliated charts whose domains cover M. Then we say that the partition $\mathcal{F}$ of M is a **foliation of dimension** n (or **codimension** $p = m - n$) if there exists on M a maximal n–foliated atlas associated with $\mathcal{F}$. We also say that $(M, \mathcal{F})$ is an n–**foliated manifold**, and $\mathcal{F}$ is an n–**foliation** of M. When we are dealing with one fixed n–foliation, we will omit "n" from names as: n–foliated chart, n–foliated atlas, etc. Each subset L_t, $t \in I$ is called a

leaf of the foliation $\mathcal{F}$. That is why a foliated chart and a foliated atlas are also named **leaf chart** and **leaf atlas** respectively.

As in the case of $\mathbb{R}^m$ we remark that the foliation $\mathcal{F}$ induces a new topology $\tau(\mathcal{F})$ on M as follows. Let $(\mathcal{U}, \varphi)$ be a foliated chart on M and L_t be a leaf of $\mathcal{F}$ such that $L_t \cap \mathcal{U} \neq \emptyset$. Suppose that a component of $L_t \cap \mathcal{U}$ is mapped by φ onto a plaque P_c^n of $\mathbb{R}^m$. Then we denote $M_c^t = \varphi^{-1}(P_c^n)$ and call it a **plaque** (**local leaf**) in M with respect to the foliation $\mathcal{F}$. Consider the topology $\tau(\mathcal{F})$ on M whose open basis consists of all plaques of M and call it the **leaf topology** of M. Clearly, any leaf L_t of $\mathcal{F}$ is clopen in M with respect to $\tau(\mathcal{F})$. Moreover, $(M, \tau(\mathcal{F}))$ becomes a manifold of dimension n with local charts $(M_c^t, \varphi|_{M_c^t})$. Note that the leaf topology $\tau(\mathcal{F})$ is finer than the original topology τ of M, and that each leaf of $\mathcal{F}$ is a connected component of $(M, \tau(\mathcal{F}))$. Also, any leaf of $\mathcal{F}$ is an n–dimensional immersed submanifold of (M, τ). However, the inclusion map of a leaf in M might be improper, that is, the inverse image of a compact subset of (M, τ) is not necessarily compact (see Example 1.5). Finally, we define an equivalence relation $\sim$ on $(M, \mathcal{F})$ as follows. We put $y \sim z$ iff y and z belong to the same leaf of $\mathcal{F}$. Thus the equivalence classes of $\sim$ are just the leaves of $\mathcal{F}$. We call the quotient space $M_{\mathcal{F}} = M/\sim$ the **leaf space** (or **space of leaves**) of $\mathcal{F}$. In some cases, $M_{\mathcal{F}}$ may have the structure of an $(m-n)$–dimensional manifold.

Now, suppose that $(\mathcal{U}, \varphi)$ is a foliated chart on the n–foliated manifold $(M, \mathcal{F})$. This means that on $\mathcal{U}$ we have the local coordinates $(x^1, ..., x^n, x^{n+1}, ..., x^{n+p})$, such that each plaque M_c^t of $\mathcal{F}$ in $\mathcal{U}$ is described by equations of the form

$$x^{n+1} = c^{n+1}, ..., x^{n+p} = c^{n+p}.$$

Thus $\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right\}$ are vector fields on $\mathcal{U}$ which are tangent to each n–dimensional submanifold M_c^t of $\mathcal{U}$. Consider another foliated chart $(\widetilde{\mathcal{U}}, \widetilde{\varphi})$ with local coordinates $(\widetilde{x}^1, ..., \widetilde{x}^n, \widetilde{x}^{n+1}, ..., \widetilde{x}^{n+p})$, and $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$. Suppose M_c^t and $M_{\tilde{c}}^t$ are two plaques in $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ respectively such that $M_c^t \cap M_{\tilde{c}}^t \neq \emptyset$. As M_c^t and $M_{\tilde{c}}^t$ are domains of some local charts on the n–dimensional submanifold L_t, we have

$$\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}, \text{ on } M_c^t \cap M_{\tilde{c}}^t. \tag{1.3}$$

As $\mathcal{U} \cap \widetilde{\mathcal{U}}$ is covered by intersections of plaques of $\mathcal{F}$, we conclude that (1.3) is true on the whole of $\mathcal{U} \cap \widetilde{\mathcal{U}}$. On the other hand, in general, on $\mathcal{U} \cap \widetilde{\mathcal{U}}$ we have

$$\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{x}^\alpha}{\partial x^i} \frac{\partial}{\partial \tilde{x}^\alpha}.$$

Taking into account (1.3) we deduce that

$$\frac{\partial \tilde{x}^\alpha}{\partial x^i} = 0, \text{ for any } \alpha \in \{n+1, ..., n+p\} \text{ and } i \in \{1, ..., n\}. \tag{1.4}$$

Thus the coordinate transformations on the n–foliated manifold $(M,\mathcal{F})$ have the following special form

$$\text{(a) } \tilde{x}^i = \tilde{x}^i(x^j, x^\beta), \qquad \text{(b) } \tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta). \tag{1.5}$$

Here, and in the sequel, we set $(x^j, x^\beta) = (x^1, ..., x^n, x^{n+1}, ..., x^{n+p})$ and $(x^\beta) = (x^{n+1}, ..., x^{n+p})$. Also, if not stated otherwise, throughout this chapter we shall use the following ranges for indices: $i, j, k, ... \in \{1, ..., n\}$; $\alpha, \beta, \gamma, ... \in \{n+1, ..., n+p\}$; $a, b, c, ... \in \{1, ..., n+p\}$.

As $\left\{\frac{\partial}{\partial x^i}\right\}$, $i \in \{1, ..., n\}$, are tangent to leaves of $\mathcal{F}$, we call $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\right\}$ an $\mathcal{F}$–**natural frame field** on $(M,\mathcal{F})$. Then the transformations of $\mathcal{F}$–natural frame fields on $(M,\mathcal{F})$ are given by (1.3) and

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial \tilde{x}^i}{\partial x^\alpha}\, \frac{\partial}{\partial \tilde{x}^i} + \frac{\partial \tilde{x}^\beta}{\partial x^\alpha}\, \frac{\partial}{\partial \tilde{x}^\beta}\,. \tag{1.6}$$

Similarly, $\{dx^i, dx^\alpha\}$ is called an $\mathcal{F}$–**natural coframe field** on $(M,\mathcal{F})$. Then by using (1.5) we deduce that the transformations of $\mathcal{F}$–natural coframe fields on $(M,\mathcal{F})$ are given by

$$\begin{aligned} &\text{(a) } d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\, dx^j + \frac{\partial \tilde{x}^i}{\partial x^\alpha}\, dx^\alpha, \\ &\text{(b) } d\tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\, dx^\beta. \end{aligned} \tag{1.7}$$

Now, we present another approach to foliations on manifolds. First, we note that an n–foliated manifold $(M,\mathcal{F})$ admits an n–distribution $\mathcal{D}$. Indeed, for any $x \in M$, we take the leaf L_t of $\mathcal{F}$ passing through x, and define $\mathcal{D}_x = T_x L_t$. We denote this distribution by $\mathcal{D}(\mathcal{F})$ and call it the **tangent distribution** to the foliation $\mathcal{F}$. If $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$ is a foliated chart on $(M,\mathcal{F})$, then $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ are tangent to $L_t \cap \mathcal{U}$ and therefore locally we have

$$\mathcal{D}(\mathcal{F}) = \text{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}.$$

Thus by assertion (iii) of Theorem 1.1.1 we deduce that the tangent distribution to a foliation is integrable. Conversely, suppose that $\mathcal{D}$ is an integrable n–distribution on M. Then by definition (see Section 1.1), for any $x \in M$ there exists a local chart $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$ on M such that all the submanifolds of $\mathcal{U}$ given by $x^\alpha = c^\alpha$, $\alpha \in \{n+1, ..., n+p\}$, are integral manifolds of $\mathcal{D}$. According to the terminology we introduced for a foliation, we are entitled to call these integral manifolds as **plaques** of $\mathcal{D}$ in M. Then consider on M a new topology $\tau(\mathcal{D})$ whose basis consists of all plaques of $\mathcal{D}$ in M. As M is covered by the set of all plaques of $\mathcal{D}$ we deduce that $(M, \tau(\mathcal{D}))$ is an n–dimensional integral manifold of $\mathcal{D}$. Moreover, it follows that any other

n–dimensional integral manifold of $\mathcal{D}$ is an open submanifold of $(M, \tau(\mathcal{D}))$. A connected component of $(M, \tau(\mathcal{D}))$ passing through $x \in M$ is called a **leaf** of $\mathcal{D}$ through x. Thus M admits a disjoint partition defined by the leaves of $\mathcal{D}$, and there exists on M an n–foliated atlas that consists of local charts covered by plaques of $\mathcal{D}$. Hence an integrable n–distribution $\mathcal{D}$ defines an n–foliation $\mathcal{F}(\mathcal{D})$ of M. Based on this discussion we may state the following.

Theorem 1.1. *Let M be an $(n+p)$–dimensional manifold. Then the following assertions are equivalent:*

(i) *There exists an n–foliation on M.*
(ii) *There exists an integrable n–distribution on M.*

Next, we discuss the relationship between involutive and integrable distributions. To achieve this in its full generality, we first refer to line fields. Let $\mathcal{D}$ be a line field on M represented locally by a vector field:

$$X = X^a(x^1, ..., x^m)\,\frac{\partial}{\partial x^a}\,, \quad \forall\,(x^a) \in \mathcal{U} \subset M. \tag{1.8}$$

According to the definition of integral manifolds of a distribution (see Section 1.1), the **integral curves** of $\mathcal{D}$ on $\mathcal{U}$ should be curves that are tangent to X. Thus they are given by the solutions of the following system of ordinary differential equations

$$\frac{dx^a}{dt} = X^a(x^1, .., x^m). \tag{1.9}$$

Then we recall the following (cf. Sternberg [Ste83], p. 372).

Theorem 1.2. (Existence and Uniqueness Theorem for ODE). *Let $f^a(t, x^b)$, $a, b \in \{1, ..., m\}$, be smooth functions defined in some neighbourhood of the origin in $\mathbb{R}^{m+1}$. Then there exist neighbourhoods, I of 0 in $\mathbb{R}$ and $\mathcal{U}$ of the origin in $\mathbb{R}^m$ such that for any $(x_0^a) \in \mathcal{U}$ and all $t \in I$ there are unique functions $u^a(t, x_0^b)$ such that*

$$\frac{du^a}{dt} = f^a(t, u^b) \quad \textit{and} \quad u^a(0, x_0^b) = x_0^a, \;\; a, b \in \{1, ..., m\}.$$

Thus applying this theorem to our system (1.9) we deduce that there exists a unique integral curve of X passing through a fixed point $x_0 = (x_0^a) \in \mathcal{U}$. Then we can state the following.

Proposition 1.3. *Any line field $\mathcal{D}$ on a manifold M is integrable.*

Taking into account Theorem 1.1 and Proposition 1.3 we obtain:

Corollary 1.4. *Any line field $\mathcal{D}$ on M defines a 1–foliation $\mathcal{F}(\mathcal{D})$ of M.*

Also, we note that a line field $\mathcal{D}$ on M is involutive, since $[X, X] = 0$, for any $X \in \Gamma(\mathcal{D})$. To obtain, in general, the relationship between integrable and involutive distributions we prove the following.

Lemma 1.5. *Let X be a smooth vector field on an open subset $\mathcal{V}$ of M, and $x_0 \in \mathcal{V}$ such that $X(x_0) \neq 0$. Then there exists a local chart $\{(\mathcal{U}, \varphi) : (x^a)\}$ about x_0, such that $X = \dfrac{\partial}{\partial x^1}$ on $\mathcal{U}$.*

Proof. As X is smooth on $\mathcal{V}$ and $X(x_0) \neq 0$, there exists a neighbourhood $\widetilde{\mathcal{U}}$ of x_0 such that $X(x) \neq 0$ for all $x \in \widetilde{\mathcal{U}}$. Thus $\mathcal{D} = \mathrm{span}\{X|_{\widetilde{\mathcal{U}}}\}$ is a line field on $\widetilde{\mathcal{U}}$. Making $\widetilde{\mathcal{U}}$ smaller, if necessary, by Corollary 1.4 there exists a 1–foliated local chart $\{(\widetilde{\mathcal{U}}, \widetilde{\varphi}) : (\tilde{x}^1, \tilde{x}^2, ..., \tilde{x}^m)\}$ about x_0. This means that $\widetilde{\mathcal{U}}$ is covered by integral curves $\widetilde{\Gamma}_c$ given by equations $\tilde{x}^\alpha = c^\alpha$, for $\alpha \in \{2, ..., m\}$. On the other hand, by assertion (iii) of Theorem 1.1.1 we may write

$$\mathcal{D} = \mathrm{span}\left\{\frac{\partial}{\partial \tilde{x}^1}\right\} \quad \text{on } \widetilde{\mathcal{U}}.$$

Hence X is expressed as follows

$$X = \widetilde{X}^1(\tilde{x}^1, \tilde{x}^2, ..., \tilde{x}^m)\, \frac{\partial}{\partial \tilde{x}^1},$$

where $\widetilde{X}^1$ is a smooth non–zero function on $\widetilde{\mathcal{U}}$. Now, on each integral curve $\widetilde{\Gamma}_c$ we define the function

$$x^1 = \int_0^{\tilde{x}^1} \frac{1}{\widetilde{X}^1(t, c^2, ..., c^m)}\, dt = f^1(\tilde{x}^1, c^2, ..., c^m).$$

Then it is easy to check that the functions

$$x^1 = f^1(\tilde{x}^1, \tilde{x}^2, ..., \tilde{x}^m),\ x^2 = \tilde{x}^2, ..., x^m = \tilde{x}^m,$$

define a new foliated chart $\{(\mathcal{U}, \varphi) : (x^1, x^2, ..., x^m)\}$ about x_0, and $X = \dfrac{\partial}{\partial x^1}$ on $\mathcal{U}$. ■

Now, suppose that $\{E_i\}$, $i \in \{1, ..., n\}$ is a system of smooth vector fields on an open subset $\mathcal{V}$ of an $(n+p)$–dimensional manifold M, $n > 1$, $p > 0$. Then we say that $\{E_i\}$ is an **Abelian** (**commutative**) **system of vector fields** if we have

$$[E_i, E_j] = 0, \quad \forall\, i, j \in \{1, ..., n\}. \tag{1.10}$$

Then Lemma 1.5 can be generalized as follows.

Lemma 1.6. *Let $\{E_i\}$ be an Abelian system of vector fields on an open subset $\mathcal{V}$ of M and $x_0 \in \mathcal{V}$ such that $\{E_i(x_0)\}$, $i \in \{1, ..., n\}$, are linearly independent.*

Then there exists a local chart $\{(\mathcal{U}, \varphi) : (x^1, ..., x^n, x^{n+1}, ..., x^{n+p})\}$ *about* x_0, *such that*

$$E_i = \frac{\partial}{\partial x^i}\,, \quad i \in \{1, ..., n\}. \tag{1.11}$$

Proof. As $\{E_1, ..., E_n\}$ are smooth on $\mathcal{V}$ and linearly independent at x_0, there exists a neighbourhood $\widetilde{\mathcal{U}}$ of x_0 on which these vector fields are linearly independent. Thus

$$\mathcal{D} = \operatorname{span}\{E_{1|\widetilde{\mathcal{U}}}, ..., E_{n|\widetilde{\mathcal{U}}}\},$$

is an n–distribution on $\widetilde{\mathcal{U}}$. First, for $n = 1$ the assertion is true by Lemma 1.5. Suppose the assertion is true for $1 < h < n$, that is, there exists a local chart $\{(\overline{\mathcal{U}}, \overline{\varphi}) : (\bar{x}^1, ..., \bar{x}^n, \bar{x}^{n+1}, ..., \bar{x}^{n+p})\}$ such that

$$E_1 = \frac{\partial}{\partial \bar{x}^1}\,, \dots, \; E_h = \frac{\partial}{\partial \bar{x}^h} \quad \text{on } \overline{\mathcal{U}}.$$

With respect to this coordinate system we set

$$E_r = \overline{E}_r^i \, \frac{\partial}{\partial \bar{x}^i} + \overline{E}_r^\alpha \, \frac{\partial}{\partial \bar{x}^\alpha}\,, \;\; \forall\, r \in \{h+1, ..., n\}.$$

Then we see that

$$[E_s, E_r] = 0, \quad \forall\, s \in \{1, ..., h\}, \; r \in \{h+1, ..., n\},$$

imply that the local components $(\overline{E}_r^a)$ of the vector field E_r, do not depend on $(\bar{x}^1, ..., \bar{x}^h)$, for any $r \in \{h+1, ..., n\}$. Now, we consider the transformation of coordinates

$$\begin{aligned} x'^s &= \bar{x}^s + f^s(\bar{x}^{h+1}, ..., \bar{x}^n, \bar{x}^{n+1}, ..., \bar{x}^{n+p}), \; s \in \{1, ..., h\}, \\ x'^r &= \bar{x}^r, \; r \in \{h+1, ..., n\}, \\ x'^\alpha &= \bar{x}^\alpha, \; \alpha \in \{n+1, ..., n+p\}, \end{aligned}$$

where f^s are solutions of the linear first order partial differential equations

$$\frac{\partial u^s}{\partial \bar{x}^r} \, \overline{E}_{h+1}^r + \frac{\partial u^s}{\partial \bar{x}^\alpha} \, \overline{E}_{h+1}^\alpha = 0, \quad s \in \{1, ..., h\}.$$

It is easy to see that with respect to the new coordinate system the vector field E_{h+1} is expressed as follows

$$E_{h+1} = \overline{E}_{h+1}^r \, \frac{\partial}{\partial x'^r} + \overline{E}_{h+1}^\alpha \, \frac{\partial}{\partial x'^\alpha}\,,$$

where the local components depend only on $(x'^{h+1}, ..., x'^n, x'^{n+1}, ..., x'^{n+p})$. Finally, apply Lemma 1.5 for E_{h+1} and obtain a coordinate system $(x^1, ..., x^n, x^{n+1}, ..., x^{n+p})$, where:

$$
\begin{aligned}
x^s &= x'^s, \ s \in \{1, ..., h\}, \\
x^r &= x^r(x'^{h+1}, ..., x'^n; x'^{n+1}, ..., x'^{n+p}), \ r \in \{h+1, ..., n\}, \\
x^\alpha &= x^\alpha(x'^{h+1}, ..., x'^n, x'^{n+1}, ..., x'^{n+p}), \ \alpha \in \{n+1, ..., n+p\},
\end{aligned}
$$

with respect to which $E_1 = \dfrac{\partial}{\partial x^1}, \ldots, E_{h+1} = \dfrac{\partial}{\partial x^{h+1}}\cdot$ This completes the proof of the lemma. ■

Now, we can give a very simple proof of a famous theorem on the geometry of distributions.

Theorem 1.7. (Frobenius Theorem). *Let M be an m–dimensional manifold and $\mathcal{D}$ an n–distribution on M with $0 < n < m$. Then the following assertions are equivalent:*

(i) *$\mathcal{D}$ is an integrable distribution.*
(ii) *$\mathcal{D}$ is an involutive distribution.*

Proof. (i) $\Longrightarrow$ (ii). Since $\mathcal{D}$ is integrable, by the assertion (iii) of Theorem 1.1.1, for any point $x \in M$ there exists a foliated chart $\{(\mathcal{U}, \varphi) : (x^1, ..., x^n, x^{n+1}, ..., x^{n+p})\}$ such that

$$\mathcal{D} = \text{span}\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}, \quad \text{on } \mathcal{U}.$$

Thus we have

$$X = X^i \, \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = Y^i \, \frac{\partial}{\partial x^i}\,,$$

for any $X, Y \in \Gamma(\mathcal{D})$. Then by direct calculations using (1.1.8) we obtain

$$[X, Y] = \left(X^j \, \frac{\partial Y^i}{\partial x^j} - Y^j \, \frac{\partial X^i}{\partial x^j}\right) \frac{\partial}{\partial x^i}\,,$$

that is, $[X, Y] \in \Gamma(\mathcal{D})$. Hence $\mathcal{D}$ is involutive.

(ii) $\Longrightarrow$ (i). Let $\{(\mathcal{U}, \varphi); (x^a)\}$ be a local chart on M. $\mathcal{D}$ being involutive, it is defined on $\mathcal{U}$ by n linearly independent vector fields $\{E_i\}$ such that

$$[E_i, E_j] = C_i{}^k{}_j \; E_k, \tag{1.12}$$

where $C_i{}^k{}_j$ are n^3 smooth functions on $\mathcal{U}$. When all $C_i{}^k{}_j$ vanish on $\mathcal{U}$, the assertion is proved by Lemma 1.6 via Theorem 1.1.1. If $\{E_i\}$ is not an Abelian system of vector fields on $\mathcal{U}$, we shall construct an Abelian one $\{\bar{E}_i\}$ as follows. We put $p = m-n$ and consider $(x^1, ..., x^n, x^{n+1}, ..., x^{n+p})$ as local coordinates on $\mathcal{U}$. Since $\{E_i\}$ are linearly independent on $\mathcal{U}$, we have

$$E_i = E_i^j \, \frac{\partial}{\partial x^j} + E_i^\alpha \, \frac{\partial}{\partial x^\alpha}, \text{ and } \operatorname{rank}[E_i^j, E_i^\alpha] = n, \quad i, j \in \{1, ..., n\}, \quad \alpha \in \{n+1, ..., n+p\}. \tag{1.13}$$

With no loss of generality we may suppose that $\det[E_i^j] \neq 0$ on $\mathcal{U}$. Then we solve the equations (1.13) with respect to $\left\{\dfrac{\partial}{\partial x^i}\right\}$ and obtain

$$\frac{\partial}{\partial x^i} = L_i^j E_j + L_i^\alpha \frac{\partial}{\partial x^\alpha}, \; i \in \{1, ..., n\}.$$

Next, consider $\mathcal{D}$ spanned on $\mathcal{U}$ by $\bar{E}_i = L_i^j E_j$, $i \in \{1, ..., n\}$. Thus we have

$$\bar{E}_i = \frac{\partial}{\partial x^i} - L_i^\alpha \frac{\partial}{\partial x^\alpha} \cdot \tag{1.14}$$

Taking into account that $\mathcal{D}$ is involutive we should have

$$[\bar{E}_i, \bar{E}_j] = \bar{C}_i{}^k{}_j \bar{E}_k. \tag{1.15}$$

Finally, using (1.14) in (1.15) and taking into account that $\left\{\dfrac{\partial}{\partial x^a}\right\}$ is a frame field on $\mathcal{U}$, we deduce that $\bar{C}_i{}^k{}_j = 0$ for any $i, j, k \in \{1, ..., n\}$. Hence $\{\bar{E}_i\}$ is an Abelian system of vector fields, and our assertion follows from Lemma 1.6 and Theorem 1.1.1. ■

Now, we combine Theorems 1.1 and 1.7 and obtain the following corollary.

Corollary 1.8. *Let M be an $(n+p)$–dimensional manifold. Then the following assertions are equivalent:*

(i) *There exists an n–foliation on M.*
(ii) *There exists an integrable n–distribution on M.*
(iii) *There exists an involutive n–distribution on M.*

Next, we suppose that $\mathcal{D}$ is an n–distribution on M locally defined by the 1–forms $\{\omega^\alpha\}$, $\alpha \in \{n+1, ..., n+p\}$, that is,

$$\Gamma(\mathcal{D}) = \{X \in \Gamma(TM) : \omega^\alpha(X) = 0, \; \forall \alpha \in \{n+1, ..., n+p\}\}. \tag{1.16}$$

Complete $\{\omega^\alpha\}$ with n local 1–forms $\{\omega^i\}$ and obtain a non–holonomic coframe field $\{\omega^i, \omega^\alpha\}$ on M. Apply the exterior differentiation operator d to each ω^α and write

$$d\omega^\alpha = \underset{(i<j)}{A_i{}^\alpha{}_j} \; \omega^i \wedge \omega^j + B_i{}^\alpha{}_\gamma \, \omega^i \wedge \omega^\gamma + \underset{(\beta<\gamma)}{C_\beta{}^\alpha{}_\gamma} \; \omega^\beta \wedge \omega^\gamma. \tag{1.17}$$

Now consider the dual non–holonomic frame field $\{E_i, E_\alpha\}$ on M. Then we have

$$\begin{aligned} d\omega^\alpha(E_i, E_j) &= E_i(\omega^\alpha(E_j)) - E_j(\omega^\alpha(E_i)) - \omega^\alpha([E_i, E_j]) \\ &= -\omega^\alpha([E_i, E_j]). \end{aligned} \tag{1.18}$$

On the other hand, from (1.17) we deduce that

$$d\omega^{\alpha}(E_i, E_j) = A_i{}^{\alpha}{}_j. \tag{1.19}$$

Clearly, $\mathcal{D}$ is involutive if and only if the right hand side in (1.18) vanishes identically. Thus from (1.17), (1.18) and (1.19) we deduce that $\mathcal{D}$ is involutive if and only if

$$d\omega^{\alpha} = (B_i{}^{\alpha}{}_{\gamma}\, \omega^i + C_{\beta}{}^{\alpha}{}_{\gamma}\, \omega^{\beta}) \wedge \omega^{\gamma}.$$

Taking into account that any line field is involutive (cf. Proposition 1.3) we state the following.

Theorem 1.9. *Let M be an $(n+p)$–dimensional manifold with $n > 1$, $p > 0$, and $\mathcal{D}$ be an n–distribution on M locally defined by the 1–forms $\{\omega^{\alpha}\}$, $\alpha \in \{n+1, ..., n+p\}$. Then $\mathcal{D}$ is involutive if and only if there exist some 1–forms θ^{α}_{γ}, such that*

$$d\omega^{\alpha} = \theta^{\alpha}_{\gamma} \wedge \omega^{\gamma}. \tag{1.20}$$

We close this section with some examples of foliations.

Example 1.1. Let M be an m–dimensional manifold, $m > 1$, and $f : M \longrightarrow \mathbb{R}$ be a smooth function on M. For any coordinate system $\{(\mathcal{U}, \varphi) : (x^a)\}$ on M consider the coordinate representative $f_{\varphi} = f \circ \varphi^{-1}$ of f. Then we say that f is without critical points on M, if with respect to any local chart on M we have

$$\text{rank}\left[\frac{\partial f_{\varphi}}{\partial x^a}\right] = 1, \quad \text{on } \mathcal{U}.$$

In this case, for any $c \in f(M)$, $f^{-1}(c)$ is a hypersurface of M. Moreover, each component of $\mathcal{U} \cap f^{-1}(c)$ is given by the equation

$$f_{\varphi}(x^1, ..., x^m) = c.$$

Without loss of generality, we may assume that $\dfrac{\partial f_{\varphi}}{\partial x^m} \neq 0$ on $\mathcal{U}$. Then on the same domain $\mathcal{U}$, we consider new coordinate functions

$$\tilde{x}^i = x^i, \;\; \tilde{x}^m = f_{\varphi}(x^1, ..., x^m), \;\; i \in \{1, ..., m-1\},$$

with respect to which any component of $\mathcal{U} \cap f^{-1}(c)$ is given by $\tilde{x}^m = c$. Thus any f without critical points defines a foliation $\mathcal{F}_f$ of M whose leaves are connected components of level hypersurfaces of f. Next, we present two particular functions which have some relevance to semi–Riemannian geometry of foliations. First, we consider the Lorentz space $\mathbb{R}^4_1$ and the x^1–axis L. Then

$$f(x^1, x^2, x^3, x^4) = x^1 - \sqrt{(x^2)^2 + (x^3)^2 + (x^4)^2},$$

is a smooth function without critical points on $M = \mathbb{R}^4_1 \backslash L$. It is easy to see that the tangent distribution $\mathcal{D}$ to the foliation $\mathcal{F}_f$ is spanned by the non–holonomic frame field

$$\{E_1 = (x^2, h, 0, 0), E_2 = (x^3, 0, h, 0), E_3 = (x^4, 0, 0, h)\},$$

where we set $h = \sqrt{(x^2)^2 + (x^3)^2 + (x^4)^2}$. Then consider

$$\xi = x^2 E_1 + x^3 E_2 + x^4 E_3,$$

and obtain $g(\xi, X) = 0$ for any $X \in \Gamma(\mathcal{D})$, where g is the Lorentz metric (1.4.10) for $m = 4$. Thus $\mathcal{D}$ is a partially null distribution and therefore all leaves of $\mathcal{F}_f$ are degenerate hypersurfaces of $\mathbb{R}^4_1$. Actually, each leaf of $\mathcal{F}_f$ is a half–cone with vertex $(c, 0, 0, 0)$ and is situated in the domain $x^1 > c$ of M. Also, we consider the Lorentz space $\mathbb{R}^3_1$ and the function

$$f(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 - (x^3)^2.$$

Then f is smooth and has no critical points on $M = \mathbb{R}^3 \backslash \{0\}$. Two of the leaves of $\mathcal{F}_f$ are the half–cones of the light–like cone

$$(x^2)^2 + (x^2)^2 - (x^3)^2 = 0, \;\; x \neq 0,$$

situated in $x^3 > 0$ and $x^3 < 0$. Thus they are degenerate hypersurfaces of M. The other leaves are hyperboloids of one sheet and hyperboloids of two sheets situated in the exterior and interior of the cone, respectively. Thus we conclude that the tangent distribution to this foliation is neither semi–Riemannian nor partially null. ■

A generalization of this type of foliations is presented in the next example.

Example 1.2. Let M and N be two manifolds of dimensions m and p respectively and $f : M \longrightarrow N$ be a smooth function. For any point $x \in M$ we consider the local charts $\{(\mathcal{U}, \varphi) : (x^a)\}$ and $\{(\mathcal{V}, \psi) : (y^\alpha)\}$ in M and N about x and $f(x)$ respectively. Let f^s, $s \in \{1, ..., p\}$ be the functions which locally represent f with respect to these local charts. Then it is easy to check that the linear mapping

$$f_{*x} : T_x M \longrightarrow T_{f(x)} N; \;\; f_{*x} \left(X^a \;\; \frac{\partial}{\partial x^a}\bigg|_x \right) = \frac{\partial f^s}{\partial x^a} \; X^a \;\; \frac{\partial}{\partial y^s}\bigg|_{f(x)},$$

is well–defined, that is, it does not depend on local charts. Thus we have a vector bundle morphism $f_* : TM \longrightarrow TN$, which is called the **differential mapping** of f. When rank $f_{*x} = r$ for a point $x \in M$, we say that f has the rank r at x. Then f is of constant rank on M, if f has the same rank for all points of M. If in particular, f is of constant rank r=p on M, then we say that

f is a **submersion**. Thus f is a submersion if and only if f_{*x} is a surjection for any $x \in M$. If y is a point in the range of the submersion f then $f^{-1}(y)$ is called the **fiber** of f over y. To discuss the differential (topological) structure on a fiber we first note that $m \geq p$. When $r = p = m$, by elementary properties of linear mappings it follows that f_{*x} is an isomorphism, and therefore f is locally a diffeomorphism. Thus for any $x \in M$ there exist two neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of x and $y = f(x)$ respectively such that $f : \mathcal{U} \longrightarrow \mathcal{V}$ is a diffeomorphism. It follows that $f^{-1}(y) \cap \mathcal{U} = \{x\}$ and therefore the induced topology on $f^{-1}(y)$ is the discrete topology which makes this case not interesting for our study. For this reason, from now on, we consider f as a submersion with $m > p$ and put $n = m - p$. If $(c^1, ..., c^p)$ are the local coordinates of a point $c \in f(M)$ and $\{(\mathcal{U}, \varphi) : (x^a)\}$ is a local chart in M about a point of $f^{-1}(c)$, then $\mathcal{U} \cap f^{-1}(c)$ is given by the equations

$$f^s(x^1, ..., x^m) = c^s, \ s \in \{1, .., p\}, \text{rank} \left[\frac{\partial f^s}{\partial x^a} \right] = p.$$

Thus $f^{-1}(c)$ is a closed imbedded submanifold of M. Without loss of generality we may suppose that $\det \left[\frac{\partial f^s}{\partial x^t} \right] \neq 0$, for $s, t \in \{1, ..., p\}$. Then we take a new coordinate system (u^i, v^s) on M, given by

$$u^i = x^i, \ v^s = f^s(x^1, ..., x^m), \ i \in \{1, ..., n\}, \ s \in \{1, ..., p\},$$

with respect to which a component of $\mathcal{U} \cap f^{-1}(c)$ is given by equations $v^s = c^s$, $s \in \{1, ..., p\}$. Thus the submersion f defines a foliation of M whose leaves are its fibers. Moreover, from the above discussion we deduce that for any $x \in M$ there exist two local charts $\{(\mathcal{U}, \varphi) : (u^i, v^s)\}$ and $\{(\mathcal{V}, \psi) : (v^s)\}$ about x and $f(x)$ respectively such that f is locally represented by a projection

$$\psi \circ f \circ \varphi^{-1}(u^1, ..., u^n; v^1, ..., v^p) = (v^1, ..., v^p). \quad ■$$

Remark 1.3. Any n–foliation of M can be locally visualized as a submersion. Indeed, if $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$ is an n–foliated chart on M then the mapping

$$F : \mathcal{U} \longrightarrow \mathbb{R}^p, \ f(x^i, x^\alpha) = x^\alpha,$$

is a submersion whose fibers are plaques of the foliation. ■

Example 1.4. Let M, N and F be manifolds of dimensions m, p and n respectively, with $m = n + p$. Then we say that $\pi : M \longrightarrow N$ is a **fiber bundle** (**fibering**) with F as **model fiber**, if for any $x \in N$ there exist an open neighbourhood $\mathcal{V}$ of x in N and a diffeomorphism $h : \pi^{-1}(\mathcal{V}) \longrightarrow F \times \mathcal{V}$ such that $p_2 \circ h = \pi$, where $p_2 : F \times \mathcal{V} \longrightarrow \mathcal{V}$ is the projection onto the second factor. It follows that π is a surjection and each fiber $\pi^{-1}(x)$ is a closed embedded submanifold of M diffeomorphic to F. We call M, N and π the **total space**, the **base space** and the **projection** of the fiber bundle, respectively. As h is a diffeomorphism and p_2 is a projection, π has a constant rank p on M. Thus π is a submersion, and therefore the total space of a fiber bundle has an n–foliation whose leaves are the components of fibers of π. In particular, we take $\mathbb{R}^n$ as model fiber and assume that the fiber $\pi^{-1}(x)$ is a vector space and $h_x : \pi^{-1}(x) \longrightarrow \mathbb{R}^n \times \{0\}$ is an isomorphism of vector spaces. In this case we call $\pi : M \longrightarrow N$ a **vector bundle** over N. The tangent distribution to the foliation determined by fibers of π is called the **vertical distribution** and it is denoted by VM. Finally, each local chart $\{(\mathcal{V}, \psi) : (x^\alpha)\}$, $\alpha \in \{n+1, ..., n+p\}$ on N determines an n–foliated local chart $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$, $i \in \{1, ..., n\}$, on M. The transformation of coordinates on M is given by

$$\tilde{x}^i = A^i_j(x^{n+1}, ..., x^{n+p})x^j, \ \tilde{x}^\alpha = \tilde{x}^\alpha(x^{n+1}, ..., x^{n+p}). \tag{1.21}$$

It is interesting to note that on the total space of a vector bundle there exists a globally defined vector field. Indeed, consider the vector field ξ locally given by

$$\xi = x^i \frac{\partial}{\partial x^i}, \tag{1.22}$$

and by using (1.21) we deduce that ξ is globally defined on M. We call ξ the **Liouville vector field** on M. A typical example of a vector bundle is the tangent bundle TN of a manifold N. In this case, the vertical distribution VTN of TN is the tangent distribution to the foliation determined by fibers of $\pi : TN \longrightarrow N$. As it is well known (see Bejancu–Farran [BF00a]) the geometry of Finsler manifolds can be fully developed via the vertical distribution. ■

Example 1.5. Let G be an m–dimensional Lie group with operation $*$ and H be an n–dimensional connected Lie subgroup of G. For any $a \in G$, the function

$$L_a : G \longrightarrow G; \ L_a(g) = a * g, \quad \forall\, g \in G,$$

is called the left translation defined by a on G. Since both L_a and $L_{a^{-1}}$ are smooth on G we conclude that L_a is a diffeomorphism of G onto itself. Thus the left coset $a * H = L_a(H)$ is an n–dimensional connected submanifold of G. Hence the set of left cosets $\{a*H\}$ determines a foliation $\mathcal{F}_H$ of G. Moreover, if H is a closed subgroup then G/H is an $(m-n)$–dimensional manifold and $\mathcal{F}_H$ is just the foliation determined by the submersion $\pi : G \longrightarrow G/H$. Now we describe an interesting particular case of this foliation. By using the canonical identification of the complex numbers space $\mathbb{C}$ with $\mathbb{R}^2$, we consider the circle S^1 of $\mathbb{R}^2$ as the set of points $\{e^{ti}\}_{t \in \mathbb{R}}$ of $\mathbb{C}$. Then S^1 becomes a Lie group with the natural operation

$$(e^{ti}, e^{si}) \longrightarrow e^{(t+s)i}.$$

Thus the 2–dimensional torus $T^2 = S^1 \times S^1$ is also a Lie group. Next, we consider a fixed irrational number λ and define the 1–dimensional submanifold

$$H = \{(e^{\lambda ti}, e^{ti})\}_{t\in\mathbb{R}},$$

of T^2. It is easy to see that H is a connected Lie subgroup of T^2. Moreover, since λ is irrational we deduce that H is dense in T^2. Thus the foliation $\mathcal{F}_H$ on T^2 is an example of foliation whose leaves are improper immersed submanifolds. ∎

Example 1.6. Let M be a smooth manifold and $\Phi : \mathbb{R}\times M \longrightarrow M$ be a smooth mapping satisfying the conditions:

(i) $\Phi(t, \Phi(s, x)) = \Phi(t + s, x)$, $\forall\, t, s \in \mathbb{R},\ x \in M$.
(ii) $\Phi(0, x) = x$, $\forall\, x \in M$.

For a fixed $t \in \mathbb{R}$, define $\Phi_t : M \longrightarrow M$, $\Phi_t(x) = \Phi(t, x)$ and from (i) and (ii) we deduce that it has Φ_{-t} as inverse function. As any Φ_t is smooth on M, we obtain a family of diffeomorphisms $\{\Phi_t\}$ of M onto itself, which is called a **one parameter group of smooth transformations** of M. Also, for a fixed $x \in M$, the map $t \longrightarrow \Phi_t(x)$ defines a smooth curve C_x passing through x. If X_x is the tangent vector to C_x at x, then $X : x \longrightarrow X_x$ is a smooth vector field on M for which C_x is the maximal integral curve through x. However, X is not necessarily non–zero on M. Indeed, if we take

$$\Phi : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2;\ \Phi(t, x^1, x^2) = e^t(x^1, x^2), \tag{1.23}$$

we obtain $X = x^1 \dfrac{\partial}{\partial x^1} + x^2 \dfrac{\partial}{\partial x^2}$ which vanishes at $(0, 0)$. If X is non–zero on M then we obtain a 1–foliation on M, which justifies the name **global flow** on M given to Φ. In this case, X is called the **infinitesimal generator** of the one parameter group $\{\Phi_t\}$ or of the global flow Φ. Also, since every integral curve of X is defined for all $t \in \mathbb{R}$, X is a complete vector field.

The converse of the above construction, in general, is not true. However, locally it is true. To state this, we give the following definition. A **local flow** around a point $x_0 \in M$ is a smooth mapping $\Phi : (-\varepsilon, \varepsilon)\times\mathcal{V} \longrightarrow M$, where $\mathcal{V}$ is a neighbourhood of x_0 and $\varepsilon > 0$, satisfying the conditions:

(i) $\Phi(t, \Phi(s, x)) = \Phi(t + s, x)$, $\forall\, x \in \mathcal{V}$, and $t, s, t + s \in (-\varepsilon, \varepsilon)$.
(ii) $\Phi(0, x) = x$, $\forall\, x \in \mathcal{V}$.

Theorem 1.10. *Let X be a vector field on M. Then there exists a local flow around any point $x_0 \in M$.*

Proof. Let $\{(\mathcal{U}, \varphi) : (x^a)\}$ be a local chart about x_0 such that $\varphi(x_0) = (0, ..., 0)$. Then we write $X = X^a \dfrac{\partial}{\partial x^a}$ on $\mathcal{U}$ and consider the system of ordinary differential equations

$$\frac{du^a}{dt} = X^a(u^1, ..., u^m), \quad a \in \{1, ..., m\}.$$

By Theorem 1.2 there exist $\varepsilon > 0$ and a neighbourhood $\mathcal{V}^* \subset \varphi(\mathcal{U})$ of the origin in $\mathbb{R}^m$, such that for any $(x^1, ..., x^m) \in \mathcal{V}^*$ the system has the unique solution $u^a(t, x^1, ..., x^m)$ satisfying

$$u^a(0, x^1, ..., x^m) = (x^1, ..., x^m). \tag{1.24}$$

Next, consider $\mathcal{V} = \varphi^{-1}(\mathcal{V}^*)$ and the smooth mapping $\Phi : (-\varepsilon, \varepsilon) \times \mathcal{V} \to M$, locally represented by

$$\Phi^a(t, x^1, ..., x^m) = u^a(t, x^1, ..., x^m). \tag{1.25}$$

Since the solutions $u^a(t + s, x^b)$ and $u^a(t, u^b(s, x^c))$ satisfy the same initial condition $(0, u^a(s, x^b))$, (i) is a consequence of the uniqueness of the solution from Theorem 1.2. Finally, (ii) follows from (1.24) and (1.25).

Taking into account Corollary 1.4 we deduce that local flows of a non–zero vector field on M determine a 1–foliation.

It is worth mentioning here that a non–singular system of ordinary differential equations, when reduced to first order, becomes a non–zero vector field. Using Theorem 1.10 above, we see that the orbits of the local flow represent the local solutions of the system, and these fit together to give a 1–foliation (see Proposition 1.3 and Corollary 1.4). ∎

Both Examples 1.5 and 1.6 are particular cases of locally free actions of Lie groups on manifolds, which we describe in the next example.

Example 1.7. Let G be an n–dimensional Lie group whose operation is denoted by $*$ and M be an m–dimensional manifold. Then we say that G acts as a **Lie transformation group** on M if there exists a smooth mapping $\Phi : G \times M \longrightarrow M$ satisfying the conditions:

(i) $\Phi(g, \Phi(h, x)) = \Phi(g * h, x), \forall\, g, h \in G, \ x \in M.$
(ii) $\Phi(e, x) = x$, $\forall\, x \in M$, where e is the unit element of G.

The **orbit** through the point $x \in M$ is the range of the smooth mapping

$$\Phi_x : G \longrightarrow M\,; \quad \Phi_x(g) = \Phi(g, x), \quad \forall\, g \in G.$$

We say that the action Φ of G on M is **locally free** if for any $x \in M$, there exists a neighbourhood $\mathcal{V}$ of e in G such that Φ_x is injective on $\mathcal{V}$. It is easy to see that Φ is locally free if and only if for each $x \in M$ the isotropy group $G_x = \{g \in G : \Phi(g, x) = x\}$ is discrete. Clearly, any orbit of a locally free action is an n–dimensional immersed submanifold of M. Moreover, in this case, all orbits of Φ determine an n–foliation of M. We note that the action in Example 1.5 is given by left translations on G and therefore $\{e\}$ is the isotropy group of any $g \in G$. On the contrary, if we consider the action (1.23), then the isotropy group of $(0,0)$ is $\mathbb{R}$, and thus that action does not determine a foliation of $\mathbb{R}^2$. ∎

Foliations can also be induced by some geometric structures on submanifolds. In the next example we present a large class of real submanifolds of a Kähler manifold which admits a totally real foliation (in a sense which is going to be defined).

Example 1.8. Let (M, J) be an **almost complex manifold**, where M is a real $2m$–dimensional manifold and J is a tensor field of type $(1,1)$ on M satisfying $J^2 = -I$ on TM. The Nijenhuis tensor field of J is a tensor field of type $(1,2)$ on M given by

$$[J,J](X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY], \tag{1.26}$$

for any $X, Y \in \Gamma(TM)$. If there exists a complex coordinate system about each point of M, and the transformations of such coordinates are holomorphic functions, then M is called a **complex manifold**. By a famous result of Newlander and Nirenberg [NN57] it is known that an almost complex manifold (M, J) is a complex manifold if and only if the Nijenhuis tensor field of J vanishes identically on M. Next we suppose that g is a **Hermitian** (**almost Hermitian**) **metric** on the complex (almost complex) manifold (M, J), that is,

$$\begin{aligned} &\text{(a) } g(JX,JY) = g(X,Y), \qquad \text{or equivalently} \\ &\text{(b) } g(X,JY) + g(JX,Y) = 0, \end{aligned} \tag{1.27}$$

for any $X, Y \in \Gamma(TM)$. Then we say that (M, J, g) is a **Hermitian** (**almost Hermitian**) **manifold**. Finally, we define the fundamental 2–form Ω of (M, J, g) by

$$\Omega(X,Y) = g(X,JY), \quad \forall\, X,Y \in \Gamma(TM). \tag{1.28}$$

If Ω is closed, that is, $d\Omega = 0$, we say that the Hermitian (almost Hermitian) manifold (M, J, g) is a **Kähler** (**almost Kähler**) **manifold**. If $\widetilde{\nabla}$ is the Levi–Civita connection on M with respect to g, then it is proved that (M, J, g) is a Kähler manifold if and only if J is parallel with respect to $\widetilde{\nabla}$, that is (cf. Yano–Kon, [YK84], p. 128),

$$(\widetilde{\nabla}_X J)(Y) = \widetilde{\nabla}_X JY - J\widetilde{\nabla}_X Y = 0, \quad \forall\, X,Y \in \Gamma(TM). \tag{1.29}$$

Now, we consider a real n–dimensional submanifold N of a Kähler manifold (M, J, g). Then we say that N is a CR–**submanifold** (**Cauchy–Riemann submanifold**) of M if there exists on N a real $2p$–dimensional distribution $\mathcal{D}$ satisfying the following conditions:

(i) $\mathcal{D}$ is a **holomorphic** (J–**invariant**) **distribution**, i.e. $J(\mathcal{D}) = \mathcal{D}$.
(ii) The complementary orthogonal distribution $\mathcal{D}^\perp$ to $\mathcal{D}$ in TN is **totally real** (J–**anti–invariant**), i.e. $J(\mathcal{D}^\perp)$ is a vector subbundle of the normal bundle $TN^\perp$ of N.

Since 1978, when the concept of CR–submanifold was introduced by Bejancu [B78], several interesting results on its differential geometry have been obtained, some of them being brought together in the monographs of Yano and Kon [YK83] and Bejancu [B86a]. When $\mathcal{D}^\perp = \{0\}$ (resp. $\mathcal{D} = \{0\}$) N becomes a **complex** (resp. **totally real**) **submanifold** of M. It is noteworthy that any real hypersurface N of M is a CR–submanifold which is neither complex nor totally real, provided $m > 1$. Indeed, in this case we define $\mathcal{D}^\perp = J(TN^\perp)$ and take $\mathcal{D}$ as complementary orthogonal distribution to $\mathcal{D}^\perp$ in TN. Since $\mathcal{D}^\perp$ is a line field on N, by Corollary 1.4 we can state the following.

Proposition 1.11. *Any real hypersurface of a real $2m$–dimensional Kähler manifold admits a totally real 1–foliation, provided $m > 1$.*

It is interesting that this result can be extended to any CR–submanifold of a Kähler manifold. To achieve this, we first use (1.5.9), (1.29), (1.27) and (1.5.8) and for any $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $Z \in \Gamma(\mathcal{D})$, we obtain

$$g(\widetilde{\nabla}_X Y, JZ) = -g(Y, J\widetilde{\nabla}_X Z) = g(JY, \widetilde{\nabla}_X Z) = g(JY, \widetilde{\nabla}_Z X).$$

Similarly, we deduce that

$$g(\widetilde{\nabla}_Y X, JZ) = g(JX, \widetilde{\nabla}_Z Y) = -g(J\widetilde{\nabla}_Z X, Y) = g(\widetilde{\nabla}_Z X, JY).$$

Then by using again (1.5.8) for $\widetilde{\nabla}$ we obtain

$$g([X, Y], JZ) = g(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X, JZ) = 0.$$

Thus $[X, Y] \in \Gamma(\mathcal{D}^\perp)$, that is, $\mathcal{D}^\perp$ is involutive. Taking into account that the leaves of $\mathcal{D}^\perp$ are totally real submanifolds we call the foliation defined by $\mathcal{D}^\perp$ a **totally real foliation** of N. When N is neither a complex submanifold nor a totally real submanifold, we say that it is a **proper** CR–**submanifold**. Then the above discussion enables us to present a new class of foliations.

Theorem 1.12. *Let N be a proper CR–submanifold of a Kähler manifold (M, J, g). Then there exists on N a totally real foliation.*

The concept of CR–submanifold has been considered by several authors for manifolds endowed with geometrical structures other than the Kählerian one. For example we mention: locally conformal symplectic structure (cf. Blair–Chen [BC79], Ornea [Orn86]), Sasakian structure (cf. Yano–Kon [YK82], Bejancu–Papaghiuc [BP81]), quaternionic Kählerian structure (cf. Barros–Chen–Urbano [BCU81], Bejancu [B86b]), etc. The integrability of $\mathcal{D}^\perp$ was first proved by Blair–Chen [BC79] for CR–submanifolds of locally conformal symplectic manifolds. ■

2.2 Adapted Tensor Fields on a Foliated Manifold

Let M be an $(n+p)$–dimensional manifold and $\mathcal{F}$ be an n–foliation of M. Denote by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$ and consider a complementary distribution $\mathcal{D}'$ to $\mathcal{D}$ in TM. As it was shown in Section 1.1, the paracompactness of M guarantees the existence of $\mathcal{D}'$. However, in this chpater, $\mathcal{D}'$ is not an intrinsic object of the foliation. Thus we may say that our study of the foliation $\mathcal{F}$ is developed with respect to a fixed transversal distribution $\mathcal{D}'$ (for terminology see Section 1.2). When a Riemannian (semi–Riemannian) metric is considered on M (this is always the case starting from Chapter 3) a canonical $\mathcal{D}'$ is defined and thus the study depends on both the foliation and the metric.

The purpose of this section is to develop a tensor calculus adapted to the decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}', \tag{2.1}$$

where $\mathcal{D}'$ is a fixed transversal distribution. To achieve this goal we first construct a local frame field adapted to (2.1) as follows. Let $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$, $i \in \{1, ..., n\}$, $\alpha \in \{n+1, ..., n+p\}$, be a foliated chart on $(M, \mathcal{F})$. Then $\mathcal{D}$ is locally represented on $\mathcal{U}$ by the natural field of frames $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$. If $\{E_{n+1}, ..., E_{n+p}\}$ locally represents $\mathcal{D}'$ on $\mathcal{U}$, then $\left\{ \frac{\partial}{\partial x^i}, E_\alpha \right\}$ is a non–holonomic frame field on $\mathcal{U}$ with respect to (2.1). Now we express each $\frac{\partial}{\partial x^\alpha}$ with respect to this frame field:

$$\frac{\partial}{\partial x^\alpha} = A^i_\alpha \, \frac{\partial}{\partial x^i} + A^\beta_\alpha \, E_\beta. \tag{2.2}$$

As the transition matrix from the non–holonomic frame field $\left\{ \frac{\partial}{\partial x^i}, E_\alpha \right\}$ to the natural frame field $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha} \right\}$ is

$$\varLambda = \begin{bmatrix} \delta^i_j & A^i_\alpha \\ 0 & A^\beta_\alpha \end{bmatrix},$$

we conclude that $[A^\beta_\alpha]$ is a non–singular matrix of functions on $\mathcal{U}$. Thus

$$\frac{\delta}{\delta x^\alpha} = A^\beta_\alpha E_\beta, \quad \alpha \in \{n+1, ..., n+p\},$$

also represent locally $\mathcal{D}'$ on $\mathcal{U}$. Then (2.2) becomes

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - A^i_\alpha \, \frac{\partial}{\partial x^i}. \tag{2.3}$$

Next, we consider another foliated chart $\{(\widetilde{\mathcal{U}},\widetilde{\varphi}) : (\widetilde{x}^i,\widetilde{x}^\alpha)\}$ such that $\mathcal{U}\cap\widetilde{\mathcal{U}}\neq\emptyset$. Then, by direct calculations using (2.3) for both charts, (1.3) and (1.6) we deduce that

$$\frac{\delta}{\delta x^\alpha}=\frac{\partial\widetilde{x}^\beta}{\partial x^\alpha}\,\frac{\delta}{\delta\widetilde{x}^\beta}\,, \tag{2.4}$$

and

$$A^j_\alpha\,\frac{\partial\widetilde{x}^i}{\partial x^j}=\widetilde{A}^i_\beta\,\frac{\partial\widetilde{x}^\beta}{\partial x^\alpha}+\frac{\partial\widetilde{x}^i}{\partial x^\alpha}\,, \tag{2.5}$$

on $\mathcal{U}\cap\widetilde{\mathcal{U}}$.

Thus, for a given $\mathcal{D}'$ there exist np functions A^i_α on $\mathcal{U}$ which satisfy (2.5) with respect to the coordinate transformations (1.5) of two foliated charts. The converse is also true. If A^i_α are functions on $\mathcal{U}$ satisfying (2.5), then define $\dfrac{\delta}{\delta x^\alpha}$ by (2.3) and obtain (2.4). Thus we obtain a distribution $\mathcal{D}'$ that is complementary to $\mathcal{D}$ in TM and locally represented by $\left\{\dfrac{\delta}{\delta x^\alpha}\right\}\cdot$ Summing up this discussion we can state the following.

Theorem 2.1. *Let $(M,\mathcal{F})$ be a foliated manifold whose tangent distribution is $\mathcal{D}$. Then there exists a complementary distribution $\mathcal{D}'$ to $\mathcal{D}$ in TM if and only if on the domain of each foliated chart on M there exist np smooth functions A^i_α satisfying* (2.5) *with respect to* (1.5).

We call $\left\{\dfrac{\partial}{\partial x^i},\dfrac{\delta}{\delta x^\alpha}\right\}$, where $\dfrac{\delta}{\delta x^\alpha}$, $\alpha\in\{n+1,...,n+p\}$ are given by (2.3) a **semi–holonomic frame field** on $\mathcal{U}$. Vector fields of the form (2.3) have been used by Reinhart [Rei59a] and Vaisman [Vai71] in their works on foliations.

In particular, if $\mathcal{D}$ is a line field on M then (2.3) becomes

$$\frac{\delta}{\delta x^\alpha}=\frac{\partial}{\partial x^\alpha}-A_\alpha\,\frac{\partial}{\partial x^1}\,,\quad \alpha\in\{2,...,m\}, \tag{2.6}$$

where A_α are $m-1$ functions on $\mathcal{U}$ satisfying

$$A_\alpha\,\frac{\partial\widetilde{x}^1}{\partial x^1}=\widetilde{A}_\beta\,\frac{\partial\widetilde{x}^\beta}{\partial x^\alpha}+\frac{\partial\widetilde{x}^1}{\partial x^\alpha}\,,\quad \alpha,\beta\in\{2,...,m\}, \tag{2.7}$$

with respect to the transformations

$$\widetilde{x}^1=\widetilde{x}^1(x^1,x^\alpha),\ \widetilde{x}^\alpha=\widetilde{x}^\alpha(x^\beta),\quad \alpha,\beta\in\{2,...,m\}. \tag{2.8}$$

Similarly, if $\mathcal{D}$ is a distribution of codimension one, then both $\mathcal{D}$ and $\mathcal{D}'$ are integrable and we may consider foliated charts whose coordinates $(x^1,...,x^{m-1},t)$ and $(\widetilde{x}^1,...,\widetilde{x}^{m-1},\widetilde{t})$ are transformed as follows

$$\widetilde{x}^i=\widetilde{x}^i(x^j),\ \widetilde{t}=\widetilde{t}(t),\quad i,j\in\{1,...,m-1\}. \tag{2.9}$$

In this case we can choose a natural frame field $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial t}\right\}$ on M such that $\dfrac{\partial}{\partial x^i} \in \Gamma(\mathcal{D})$ and $\dfrac{\partial}{\partial t} \in \Gamma(\mathcal{D}')$.

Now, we come back to the general case and consider the dual vector bundles $\mathcal{D}^*$ and $\mathcal{D}^{'*}$ to $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then an **adapted tensor field** of type $(q, s; r, t)$ on the foliated manifold $(M, \mathcal{F})$ is an $F(M) - (q + r + s + t)$–multilinear mapping

$$T : \Gamma(\mathcal{D}^*)^q \times \Gamma(\mathcal{D}^{'*})^r \times \Gamma(\mathcal{D})^s \times \Gamma(\mathcal{D}')^t \longrightarrow F(M).$$

In order to define the local components of T we consider the **dual semi–holonomic frame field** $\{\delta x^i, dx^\alpha\}$ to $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\alpha}\right\}$, where we set

$$\delta x^i = dx^i + A^i_\alpha dx^\alpha. \tag{2.10}$$

Thus, locally T is given by $n^{q+s} \cdot p^{r+t}$ smooth functions

$$\begin{aligned} &T^{i_1 \dots i_q\, \alpha_1 \dots \alpha_r}_{j_1 \dots j_s\, \beta_1 \dots \beta_t}(x^i, x^\alpha) \\ &= T\left(\delta x^{i_1}, ..., \delta x^{i_q}, dx^{\alpha_1}, ..., dx^{\alpha_r}, \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}}, \frac{\delta}{\delta x^{\beta_1}}, \dots, \frac{\delta}{\delta x^{\beta_t}}\right). \end{aligned} \tag{2.11}$$

Next, by direct calculations using (2.10), (1.7) and (2.5) we obtain

$$\text{(a) } \delta \widetilde{x}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}\, \delta x^j, \text{ and (b) } d\widetilde{x}^\alpha = \frac{\partial \widetilde{x}^\alpha}{\partial x^\beta}\, dx^\beta, \tag{2.12}$$

with respect to the coordinate transformations (1.5) on $(M, \mathcal{F})$. Then, taking into account (2.11), (2.12), (1.3) and (2.4) we state the following.

Theorem 2.2. *Let $(M, \mathcal{F})$ be a foliated manifold with transversal distribution $\mathcal{D}'$. Then there exists on M an adapted tensor field of type $(q, s; r, t)$ if and only if on the domain of each foliated chart on M there exist $n^{q+s} \cdot p^{r+t}$ smooth functions $T^{i_1 \dots i_q\, \alpha_1 \dots \alpha_r}_{j_1 \dots j_s\, \beta_1 \dots \beta_t}$ satisfying*

$$\begin{aligned} &\widetilde{T}^{k_1 \dots k_q\, \gamma_1 \dots \gamma_r}_{h_1 \dots h_s\, \varepsilon_1 \dots \varepsilon_t} \frac{\partial \widetilde{x}^{h_1}}{\partial x^{j_1}} \cdots \frac{\partial \widetilde{x}^{h_s}}{\partial x^{j_s}} \frac{\partial \widetilde{x}^{\varepsilon_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \widetilde{x}^{\varepsilon_t}}{\partial x^{\beta_t}} \\ &= T^{i_1 \dots i_q\, \alpha_1 \dots \alpha_r}_{j_1 \dots j_s\, \beta_1 \dots \beta_t} \frac{\partial \widetilde{x}^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial \widetilde{x}^{k_q}}{\partial x^{i_q}} \frac{\partial \widetilde{x}^{\gamma_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \widetilde{x}^{\gamma_r}}{\partial x^{\alpha_r}}, \end{aligned} \tag{2.13}$$

with respect to (1.5).

Remark 2.1.

(i) It is easy to check that any $F(M) - (s + t)$–multilinear mapping

$$T : \Gamma(\mathcal{D})^s \times \Gamma(\mathcal{D}')^t \longrightarrow \Gamma(\mathcal{D}),$$

defines an adapted tensor field of type $(1, s; 0, t)$ and viceversa.

(ii) Similarly, any $F(M)-(s+t)$–multilinear mapping

$$T : \Gamma(\mathcal{D})^s \times \Gamma(\mathcal{D}')^t \longrightarrow \Gamma(\mathcal{D}'),$$

defines an adapted tensor field of type $(0,s;1,t)$ and viceversa. ■

Remark 2.2. The order of indices (first the latin and then the greek indices) is not necessarily the same throughout the book. As an example the n^2p^2 functions $T^{\alpha i}_{\beta j}$ satisfying

$$\widetilde{T}^{\gamma k}_{\varepsilon h} \frac{\partial \widetilde{x}^{\varepsilon}}{\partial x^{\beta}} \frac{\partial \widetilde{x}^{h}}{\partial x^{j}} = T^{\alpha i}_{\beta j} \frac{\partial \widetilde{x}^{\gamma}}{\partial x^{\alpha}} \frac{\partial \widetilde{x}^{k}}{\partial x^{i}},$$

define an adapted tensor field of type $(1,1;1,1)$ on $(M,\mathcal{F})$. ■

In particular, an adapted tensor field of type $(0,s;0,0)$:

$$\omega : \Gamma(\mathcal{D})^s \longrightarrow F(M),$$

satisfying

$$\omega(X_{\sigma(1)},...,X_{\sigma(s)}) = \varepsilon(\sigma)\omega(X_1,...,X_s),$$

for any permutation σ of $\{1,...,s\}$, where $\varepsilon(\sigma)=\pm 1$ is the signature of σ, is called a **structural** s**–form** on $(M,\mathcal{F})$. Similarly, an adapted tensor field of type $(0,0;0,t)$:

$$\Omega : \Gamma(\mathcal{D}')^t \longrightarrow F(M),$$

which satisfies

$$\Omega(Y_{\sigma(1)},...,Y_{\sigma(t)}) = \varepsilon(\sigma)\Omega(Y_1,...,Y_t),$$

is called a **transversal** t**–form** on $(M,\mathcal{F})$. It is easy to see that any structural 1–form (resp. transversal 1–form) is a section of $\mathcal{D}^*$ (resp. $\mathcal{D}^{'*}$). We also call a section of $\mathcal{D}$ (resp. $\mathcal{D}'$) a **structural** (resp. **transversal**) **vector field** on the foliated manifold $(M,\mathcal{F})$. We can extend this terminology to the general case of adapted tensor fields as follows. We say that an adapted tensor field T is a **structural** (resp. **transversal**) **tensor field** if it is locally represented by functions $T^{i_1...i_q}_{j_1...j_s}$ (resp. $T^{\alpha_1...\alpha_r}_{\beta_1...\beta_t}$). By direct calculations, using (2.13) for a transversal tensor field, and (1.3), we deduce that

$$\frac{\partial \widetilde{T}^{\gamma_1...\gamma_r}_{\varepsilon_1...\varepsilon_t}}{\partial \widetilde{x}^k} \frac{\partial \widetilde{x}^k}{\partial x^i} \frac{\partial \widetilde{x}^{\varepsilon_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \widetilde{x}^{\varepsilon_t}}{\partial x^{\beta_t}} = \frac{\partial T^{\alpha_1...\alpha_r}_{\beta_1...\beta_t}}{\partial x^i} \frac{\partial \widetilde{x}^{\gamma_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \widetilde{x}^{\gamma_r}}{\partial x^{\alpha_r}}.$$

This enables us to give the following definition. We say that a transversal tensor field $T=(T^{\alpha_1...\alpha_r}_{\beta_1...\beta_t})$ is **basic** if we have

$$\frac{\partial T^{\alpha_1...\alpha_r}_{\beta_1...\beta_t}}{\partial x^i} = 0, \tag{2.14}$$

for any $i \in \{1, ..., n\}$ and $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_t \in \{n+1, ..., n+p\}$, with respect to any foliated chart $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$ on $(M, \mathcal{F})$.

In particular, a transversal t–form $\Omega = (\Omega_{\beta_1 \dots \beta_t})$ is basic if and only if

$$\frac{\partial \Omega_{\beta_1 \dots \beta_t}}{\partial x^i} = 0, \quad \forall\, i \in \{1, ..., n\}, \ \beta_1, ..., \beta_t \in \{n+1, ..., n+p\}, \tag{2.15}$$

on the domain of any foliated chart. It is easy to see that a t–form Ω on M is transversal if and only if

$$\Omega(X, Y_1, ..., Y_{t-1}) = 0, \ \forall\, X \in \Gamma(\mathcal{D}), \ Y_1, ..., Y_{t-1} \in \Gamma(\mathcal{D}'). \tag{2.16}$$

A t–form Ω on $(M, \mathcal{F})$ which satisfies both (2.15) and (2.16) is called basic by Reinhart [Rei83], p. 171. As the exterior differential of a basic form is basic too, a cohomology theory of basic forms has been developed (cf. Reinhart [Rei59b]). Similarly, a function f on M is called a **basic function** if it depends on (x^α) alone, that is, f is constant on each leaf of $\mathcal{F}$.

Next, we consider the projection morphisms Q and Q' of TM on $\mathcal{D}$ and $\mathcal{D}'$ respectively and state the following.

Lemma 2.3. *The mapping $T : \Gamma(\mathcal{D}') \times \Gamma(\mathcal{D}') \longrightarrow \Gamma(\mathcal{D})$ given by*

$$T(Q'X, Q'Y) = Q[Q'X, Q'Y], \quad \forall\, X, Y \in \Gamma(TM), \tag{2.17}$$

defines an adapted tensor field on $(M, \mathcal{F})$ of type $(1, 0; 0, 2)$.

Proof. First, from (2.17) it follows that T is an $F(M)$–bilinear mapping. Then take $s = 0$ and $t = 2$ in assertion (i) of Remark 2.1 and deduce that T is an adapted tensor field of type $(1, 0; 0, 2)$. ∎

Lemma 2.4. *Let $\left\{ \frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha} \right\}$ be a semi–holonomic frame field on $(M, \mathcal{F})$. Then we have*

$$\left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta} \right] = T_\alpha{}^i{}_\beta \, \frac{\partial}{\partial x^i}, \tag{2.18}$$

where we set

$$T_\alpha{}^i{}_\beta = \frac{\delta A^i_\alpha}{\delta x^\beta} - \frac{\delta A^i_\beta}{\delta x^\alpha}. \tag{2.19}$$

Proof. By direct calculations using (2.3) and elementary properties of Lie bracket we obtain (2.18). ∎

Finally, from (2.18) and (2.17) we see that $T_\alpha{}^i{}_\beta$ are the local components of the adapted tensor field T, that is, we have

$$T\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta} \right) = T_\alpha{}^i{}_\beta \, \frac{\partial}{\partial x^i}. \tag{2.20}$$

We call T the **integrability tensor** of the transversal distribution $\mathcal{D}'$. From (2.17) we see that $\mathcal{D}'$ is integrable if and only if T vanishes identically on M, which justifies the above name for T.

In the next section we will see that both torsion and curvature tensor fields of an adapted connection are determined by adapted tensor fields. As adapted connections play an important role in studying foliations, we consider adapted tensor fields as a need for this type of geometry.

2.3 Structural and Transversal Linear Connections

In the first part of this section we develop a general theory of linear connections on vector bundles over foliated manifolds. We show that two types of covariant derivatives are naturally defined by a linear connection on a vector bundle: the structural and the transversal covariant derivatives. Then we apply this theory to the structural and transversal distributions to a foliation and obtain the local components of curvature and torsion tensor fields of both the structural and transversal connections.

Let $\mathcal{F}$ be an n–foliation on the $(n+p)$–dimensional manifold M and $\mathcal{D}$ be the tangent distribution (structural distribution) to $\mathcal{F}$. Throughout this section we suppose that $\mathcal{D}'$ is a transversal distribution to $\mathcal{F}$ locally defined by the functions $\{A^i_\alpha\}$, $i \in \{1,...,n\}$, $\alpha \in \{n+1,...,n+p\}$ satisfying (2.5). Then on the domain of a foliated chart $\{(\mathcal{U},\varphi) : (x^i, x^\alpha)\}$ we consider the semi–holonomic frame field $\left\{ \dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\alpha} \right\}$, where $\dfrac{\delta}{\delta x^\alpha}$ are given by (2.3).

Next, we consider a vector bundle E of rank h over $(M, \mathcal{F})$, that is, the dimension of each fiber of E is h. Let ∇ be a linear connection on E and $\{S_a\}$, $a \in \{1,...,h\}$ be a basis of $\Gamma(E)$ on $\mathcal{U}$. Then we put

$$\text{(a) } \nabla_{\frac{\delta}{\delta x^\alpha}} S_a = F_a{}^b{}_\alpha \, S_b \quad \text{and} \quad \text{(b) } \nabla_{\frac{\partial}{\partial x^i}} S_a = C_a{}^b{}_i S_b, \tag{3.1}$$

where $\{F_a{}^b{}_\alpha, C_a{}^b{}_i\}$, $a, b \in \{1,...,h\}$, $i \in \{1,...,n\}$, $\alpha \in \{n+1,...,n+p\}$ are smooth functions on $\mathcal{U}$. Two local bases $\{S_a\}$ and $\{\tilde{S}_a\}$ of $\Gamma(E)$ are related by

$$S_a = S^b_a \tilde{S}_b, \tag{3.2}$$

where S^b_a are smooth functions on the common domain of two foliated charts on M. Then by direct calculations using (1.3), (2.4), (3.2) and (3.1) we deduce that the local coefficients of ∇ satisfy the following with respect to (1.5) and (3.2):

$$F_a{}^b{}_\alpha S^c_b = \tilde{F}_d{}^c{}_\beta S^d_a \, \frac{\partial \tilde{x}^\beta}{\partial x^\alpha} + \frac{\delta S^c_a}{\delta x^\alpha} \, , \tag{3.3}$$

$$C_a{}^b{}_i S^c_b = \tilde{C}_d{}^c{}_j S^d_a \, \frac{\partial \tilde{x}^j}{\partial x^i} + \frac{\partial S^c_a}{\partial x^i} \, . \tag{3.4}$$

Here we use the indices $a, b, \ldots \in \{1, \ldots, h\}$, $i, j, \ldots \in \{1, \ldots, n\}$, $\alpha, \beta, \ldots \in \{n+1, \ldots, n+p\}$. Conversely, suppose that on the domain of each foliated chart on $(M, \mathcal{F})$ there exist smooth functions $\{F_a{}^b{}_\alpha, C_a{}^b{}_i\}$ satisfying (3.3) and (3.4) with respect to (1.5) and (3.2). Then for any $Y = Y^\alpha \dfrac{\delta}{\delta x^\alpha}$, $X = X^i \dfrac{\partial}{\partial x^i}$ and $Z = Z^a S_a$, we define

$$\text{(a) } \nabla_Y Z = Y^\alpha Z^a{}_{|\alpha} S_a \quad \text{and} \quad \text{(b) } \nabla_X Z = X^i Z^a{}_{\|i} S_a, \tag{3.5}$$

where we set

$$\text{(a) } Z^a{}_{|\alpha} = \frac{\delta Z^a}{\delta x^\alpha} + Z^b F_b{}^a{}_\alpha \quad \text{and} \quad \text{(b) } Z^a{}_{\|i} = \frac{\partial Z^a}{\partial x^i} + Z^b C_b{}^a{}_i. \tag{3.6}$$

Extend ∇ by linearity to any vector field on M and by using (3.3)–(3.6) we deduce that ∇ is a linear connection on the vector bundle E. Thus we may state the following.

Theorem 3.1. *Let $(M, \mathcal{F})$ be a foliated manifold with structural and transversal distributions $\mathcal{D}$ and $\mathcal{D}'$, and E be a vector bundle over M. Then there exists a linear connection on E if and only if on the domain of each foliated chart on M there exist real smooth functions $\{F_a{}^b{}_\alpha, C_a{}^b{}_i\}$ satisfying* (3.3) *and* (3.4) *with respect to* (1.5) *and* (3.2).

We call $Z^a{}_{|\alpha}$ and $Z^a{}_{\|i}$ given by (3.6a) and (3.6b) respectively the **transversal covariant derivative** and **structural covariant derivative** of the section Z.

In particular, we suppose that ∇' and ∇ are linear connections on distributions $\mathcal{D}'$ and $\mathcal{D}$ respectively. Then by using the semi–holonomic frame field $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\alpha}\right\}$ induced by $\mathcal{D}'$, we put:

$$\text{(a) } \nabla'_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F'_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma}, \text{ (b) } \nabla'_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = C'_\alpha{}^\gamma{}_i \frac{\delta}{\delta x^\gamma}, \tag{3.7}$$

$$\text{(a) } \nabla_{\frac{\delta}{\delta x^\beta}} \frac{\partial}{\partial x^i} = F_i{}^k{}_\beta \frac{\partial}{\partial x^k}, \text{ (b) } \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = C_i{}^k{}_j \frac{\partial}{\partial x^k}. \tag{3.8}$$

We call ∇' (resp. ∇) a **transversal** (resp. **structural**) **linear connection** of the foliation $\mathcal{F}$ on M. Now, we take in turn $\mathcal{D}'$ and $\mathcal{D}$ instead of E from Theorem 3.1 and obtain the following.

Theorem 3.2. *Let $(M, \mathcal{F})$ be a foliated manifold with structural and transversal distributions $\mathcal{D}$ and $\mathcal{D}'$. Then we have the following assertions:*

(i) *There exists a transversal linear connection* ∇' *of* $\mathcal{F}$ *if and only if on the domain of each foliated chart on* M *there exist real smooth functions* $\{F'{}_\alpha{}^\gamma{}_\beta, C'{}_\alpha{}^\gamma{}_i\}$ *satisfying*

$$F'{}_\alpha{}^\gamma{}_\beta \, \frac{\partial \tilde{x}^\varepsilon}{\partial x^\gamma} = \tilde{F}'{}_\mu{}^\varepsilon{}_\nu \, \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \, \frac{\partial \tilde{x}^\nu}{\partial x^\beta} + \frac{\partial^2 \tilde{x}^\varepsilon}{\partial x^\alpha \partial x^\beta}\,, \tag{3.9}$$

$$C'{}_\alpha{}^\gamma{}_i \, \frac{\partial \tilde{x}^\varepsilon}{\partial x^\gamma} = \tilde{C}'{}_\beta{}^\varepsilon{}_j \, \frac{\partial \tilde{x}^\beta}{\partial x^\alpha} \, \frac{\partial \tilde{x}^j}{\partial x^i}\,, \tag{3.10}$$

with respect to (1.5).

(ii) *There exists a structural linear connection* ∇ *of* $\mathcal{F}$ *if and only if on the domain of each foliated chart on* M *there exist real smooth functions* $\{F_i{}^k{}_\alpha, C_i{}^k{}_j\}$ *satisfying*

$$F_i{}^k{}_\alpha \, \frac{\partial \tilde{x}^j}{\partial x^k} = \tilde{F}_h{}^j{}_\beta \, \frac{\partial \tilde{x}^h}{\partial x^i} \, \frac{\partial \tilde{x}^\beta}{\partial x^\alpha} + \frac{\delta}{\delta x^\alpha}\left(\frac{\partial \tilde{x}^j}{\partial x^i}\right), \tag{3.11}$$

$$C_i{}^k{}_j \, \frac{\partial \tilde{x}^h}{\partial x^k} = \tilde{C}_r{}^h{}_t \, \frac{\partial \tilde{x}^r}{\partial x^i} \, \frac{\partial \tilde{x}^t}{\partial x^j} + \frac{\partial^2 \tilde{x}^h}{\partial x^i \partial x^j}\,, \tag{3.12}$$

with respect to (1.5).

According to the terminology from Section 1.2, a linear connection ∇^* on M is called an adapted linear connection with respect to the decomposition (2.1) if and only if (1.2.1) and (1.2.2) are satisfied. Moreover, a pair (∇, ∇'), where ∇ and ∇' are linear connections on $\mathcal{D}$ and $\mathcal{D}'$ respectively determines an adapted linear connection ∇^* and viceversa (cf. Theorem 1.2.1). Thus from Theorem 3.2 we deduce the following corollary.

Corollary 3.3. *Let* $(M, \mathcal{F})$ *be a foliated manifold with structural and transversal distributions* $\mathcal{D}$ *and* $\mathcal{D}'$. *Then there exist on* M *an adapted linear connection* ∇^* *if and only if on the domain of each foliated chart on* M *there exist real smooth functions* $\{F_i{}^k{}_\alpha, C_i{}^k{}_j, F'{}_\alpha{}^\gamma{}_\beta, C'{}_\alpha{}^\gamma{}_i\}$ *satisfying* (3.9)–(3.12) *with respect to* (1.5).

Next, we consider a structural vector field $X = X^i \dfrac{\partial}{\partial x^i}$ and from (3.6) we deduce that its transversal and structural covariant derivatives with respect to ∇ on $\mathcal{D}$ are given by

$$X^i{}_{|\alpha} = \frac{\delta X^i}{\delta x^\alpha} + X^j F_j{}^i{}_\alpha, \tag{3.13}$$

and

$$X^i{}_{\|j} = \frac{\partial X^i}{\partial x^j} + X^k C_k{}^i{}_j, \tag{3.14}$$

respectively. Similarly, the transversal and structural covariant derivatives of a transversal vector field $Y = Y^\alpha \dfrac{\delta}{\delta x^\alpha}$ with respect to ∇' on $\mathcal{D}'$ are given by

$$Y^{\alpha}{}_{|\beta} = \frac{\delta Y^{\alpha}}{\delta x^{\beta}} + Y^{\gamma} F'{}_{\gamma}{}^{\alpha}{}_{\beta}, \tag{3.15}$$

and

$$Y^{\alpha}{}_{\|i} = \frac{\partial Y^{\alpha}}{\partial x^{i}} + Y^{\beta} C'{}_{\beta}{}^{\alpha}{}_{i}. \tag{3.16}$$

Now, we consider an adapted linear connection $\nabla^* = (\nabla, \nabla')$ on $(M, \mathcal{F})$ locally given by the functions $\{F_i{}^k{}_\alpha, C_i{}^k{}_j, F'{}_\alpha{}^\gamma{}_\beta, C'{}_\alpha{}^\gamma{}_i\}$ and an adapted tensor field T of type $(q, s; r, t)$ with local components $T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t}$. Then the **transversal covariant derivative** of T with respect to ∇^* is defined by

$$\begin{aligned} T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t | \gamma} &= \frac{\delta T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t}}{\delta x^{\gamma}} \\ &+ \sum_{a=1}^{q} T^{i_1 \ldots h i_{a+1} \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t} F_h{}^{i_a}{}_{\gamma} + \sum_{b=1}^{r} T^{i_1 \ldots i_q\, \alpha_1 \ldots \varepsilon \alpha_{b+1} \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t} F'{}_{\varepsilon}{}^{\alpha_b}{}_{\gamma} \\ &- \sum_{c=1}^{s} T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots h j_{c+1} \ldots j_s\, \beta_1 \ldots \beta_t} F_{j_c}{}^{h}{}_{\gamma} - \sum_{d=1}^{t} T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \varepsilon \beta_{d+1} \ldots \beta_t} F'{}_{\beta_d}{}^{\varepsilon}{}_{\gamma}. \end{aligned} \tag{3.17}$$

Similarly, we define the **structural covariant derivative** of the adapted tensor field T with respect to ∇^* by

$$\begin{aligned} T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t \| k} &= \frac{\partial T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t}}{\partial x^{k}} \\ &+ \sum_{a=1}^{q} T^{i_1 \ldots h i_{a+1} \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t} C_h{}^{i_a}{}_{k} + \sum_{b=1}^{r} T^{i_1 \ldots i_q\, \alpha_1 \ldots \varepsilon \alpha_{b+1} \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \beta_t} C'{}_{\varepsilon}{}^{\alpha_b}{}_{k} \\ &- \sum_{c=1}^{s} T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots h j_{c+1} \ldots j_s\, \beta_1 \ldots \beta_t} C_{j_c}{}^{h}{}_{k} - \sum_{d=1}^{t} T^{i_1 \ldots i_q\, \alpha_1 \ldots \alpha_r}_{j_1 \ldots j_s\, \beta_1 \ldots \varepsilon \beta_{d+1} \ldots \beta_t} C'{}_{\beta_d}{}^{\varepsilon}{}_{k}. \end{aligned} \tag{3.18}$$

In particular, for a structural 1−form $\omega = \omega_i \delta x^i$ we have:

$$\text{(a) } \omega_{i|\alpha} = \frac{\delta \omega_i}{\delta x^{\alpha}} - \omega_j F_i{}^j{}_\alpha \quad \text{and} \quad \text{(b) } \omega_{i\|j} = \frac{\partial \omega_i}{\partial x^{j}} - \omega_k C_i{}^k{}_j. \tag{3.19}$$

Similarly, for a transversal 1−form $\theta = \theta_\alpha dx^\alpha$ we obtain:

$$\text{(a) } \theta_{\alpha|\beta} = \frac{\delta \theta_\alpha}{\delta x^{\beta}} - \theta_\gamma F'{}_\alpha{}^\gamma{}_\beta \quad \text{and} \quad \text{(b) } \theta_{\alpha\|i} = \frac{\partial \theta_\alpha}{\partial x^{i}} - \theta_\gamma C'{}_\alpha{}^\gamma{}_i. \tag{3.20}$$

Remark 3.1. It is noteworthy that both covariant derivatives given by (3.17) and (3.18) define adapted tensor fields of type $(q, s; r, t+1)$ and $(q, s+1; r, t)$ respectively. ∎

Next, in order to obtain the local components of curvature and torsion tensor fields of both ∇' and ∇ we state the following.

Lemma 3.4. *Let* $\left\{ \frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha} \right\}$ *be a semi–holonomic frame field on a domain of a foliated chart on* $(M, \mathcal{F})$*, where* $\frac{\delta}{\delta x^\alpha}$ *is given by* (2.3) *for any* $\alpha \in \{n+1, ..., n+p\}$*. Then we have:*

$$\left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^i} \right] = \frac{\partial A^j_\alpha}{\partial x^i} \frac{\partial}{\partial x^j}. \tag{3.21}$$

Proof. It follows by using (2.3) and properties of the Lie bracket. ■

An interesting geometric property of the non–holonomic basis $\left\{ \frac{\delta}{\delta x^\alpha} \right\}$ of $\Gamma(\mathcal{D}')$ follows from (3.21). To state this we first give the following definition. Let X be a vector field on an open subset $\mathcal{V}$ of M and $\Phi_t : \mathcal{V}' \longrightarrow \mathcal{V}$ be the local flow of X around $x \in \mathcal{V}$. Then we say that the foliation $\mathcal{F}$ is **invariant** with respect to the action of Φ_t if for any leaf L with $L \cap \mathcal{V}' \neq \emptyset$ we have

$$\Phi_t(L \cap \mathcal{V}') \subset L', \tag{3.22}$$

where L' is also a leaf of $\mathcal{F}$.

Now, we prove the following.

Lemma 3.5. *Let* $(M, \mathcal{F})$ *be a foliated manifold and* X *a vector field on an open subset* $\mathcal{V}$ *of* M*. Then the foliation* $\mathcal{F}$ *is invariant with respect to the actions of all local flows of* X *if and only if*

$$[X, Y] \in \Gamma(\mathcal{D}_{|\mathcal{V}}), \quad \forall Y \in \Gamma(\mathcal{D}_{|\mathcal{V}}). \tag{3.23}$$

Proof. Let Φ_t be the local flow of X around $x \in \mathcal{V}$. If Φ_t satisfies (3.22) then $\mathcal{D}_{|\mathcal{V}}$ is invariant with respect to Φ_{t*}. Then we use the following formula for Lie bracket at a point (cf. O'Neill [O83], p. 31)

$$[X, Y]_x = \lim_{t \to 0} \frac{1}{t} \left(\Phi_{-t*}(Y_{\Phi_t(x)}) - Y_x \right),$$

and obtain $[X, Y]_x \in \mathcal{D}_x$. Conversely, suppose that (3.23) is satisfied and consider a foliated chart $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$ of M such that $\mathcal{U} \subset \mathcal{V}$. Then with respect to the natural field of frames $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha} \right\}$ we have

$$X_{|\mathcal{U}} = X'^i \frac{\partial}{\partial x^i} + X'^\alpha \frac{\partial}{\partial x^\alpha}.$$

Taking into account that

$$\left[X_{|\mathcal{U}}, \frac{\partial}{\partial x^i}\right] \in \Gamma(\mathcal{D}_{|\mathcal{U}}),$$

we deduce that $X^{'\alpha}$ do not depend on $(x^1, ..., x^n)$. Next, we consider the system of differential equations

$$\frac{dx^i}{dt} = X^{'i}(x^j, x^\beta),\ i, j \in \{1, ..., n\},$$
$$\frac{dx^\alpha}{dt} = X^{'\alpha}(x^\beta), \quad \alpha, \beta \in \{n+1, ..., n+p\},$$

whose solutions define local flows of X. If $(x_0^i, x_0^\alpha) \in \mathcal{U}$ is an initial condition, then we have

$$\Phi_t(x_0^i, x_0^\alpha) = (x^j(t, x_0^i, x_0^\alpha), x^\beta(t, x_0^i, x_0^\alpha)).$$

From the last p equations in the above system we deduce that $x^\beta(t, x_0^i, x_0^\alpha)$, $\beta \in \{n+1, ..., n+p\}$ do not depend on (x_0^i). Hence Φ_t carries the plaque $x^\alpha = x_0^\alpha$ to the plaque $x^\beta = x^\beta(t, x_0^\alpha)$, which completes the proof of the lemma. ■

Next, we consider the projection morphisms Q and Q' of TM to $\mathcal{D}$ and $\mathcal{D}'$ with respect to (2.1) and write $X \in \Gamma(TM)$ as follows

$$X = QX + Q'X. \tag{3.24}$$

Then we call QX (resp. $Q'X$) the **structural** (resp. **transversal**) **component** of X. Now we state the following interesting characterization of invariant foliations.

Lemma 3.6. *Let* $(M, \mathcal{F})$ *and* X *be as in Lemma* 3.5. *Then* $\mathcal{F}$ *is invariant with respect to the actions of all local flows of* X *if and only if the transversal component of* X *is basic.*

Proof. Consider a foliated chart and write X with respect to the semi–holonomic frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ as follows

$$X = X^i\ \frac{\partial}{\partial x^i} + X^\alpha\ \frac{\delta}{\delta x^\alpha}\cdot \tag{3.25}$$

Then by direct calculations using (3.21) and (3.25) we obtain

$$\left[X, \frac{\partial}{\partial x^j}\right] = \left(X^\alpha\ \frac{\partial A_\alpha^k}{\partial x^j} - \frac{\partial X^k}{\partial x^j}\right) \frac{\partial}{\partial x^k} - \frac{\partial X^\alpha}{\partial x^j}\ \frac{\delta}{\delta x^\alpha}\cdot \tag{3.26}$$

Thus the assertion follows from (3.26) by using Lemma 3.5. ■

Taking into account Lemmas 3.4 and 3.5 we obtain the following.

Theorem 3.7. *Let $\mathcal{F}$ be a foliation of M and $\mathcal{D}'$ be a transversal distribution to $\mathcal{F}$. Then $\mathcal{F}$ is invariant with respect to local flows of all non–holonomic vector fields* $\dfrac{\delta}{\delta x^\alpha}$, $\alpha \in \{n+1, ..., n+p\}$, *given by* (2.3).

Next we consider an adapted linear connection $\nabla^* = (\nabla, \nabla')$ on $(M, \mathcal{F})$ and denote by R^*, R and R' the curvature tensor fields of ∇^*, ∇ and ∇' respectively. Then we have

$$R^*(X,Y)Z = R(X,Y)QZ + R'(X,Y)Q'Z, \tag{3.27}$$

$$R(X,Y)QZ = \nabla_X \nabla_Y QZ - \nabla_Y \nabla_X QZ - \nabla_{[X,Y]} QZ, \tag{3.28}$$

$$R'(X,Y)Q'Z = \nabla'_X \nabla'_Y Q'Z - \nabla'_Y \nabla'_X Q'Z - \nabla'_{[X,Y]} Q'Z, \tag{3.29}$$

for any $X, Y, Z \in \Gamma(TM)$. Take a semi–holonomic frame field $\left\{ \dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\alpha} \right\}$ and put:

$$\begin{aligned}
&\text{(a) } R^*\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right)\frac{\partial}{\partial x^i} = R\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right)\frac{\partial}{\partial x^i} = R_i{}^h{}_{\alpha\beta}\,\frac{\partial}{\partial x^h},\\
&\text{(b) } R^*\left(\frac{\partial}{\partial x^k}, \frac{\delta}{\delta x^\alpha}\right)\frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\delta}{\delta x^\alpha}\right)\frac{\partial}{\partial x^i} = R_i{}^h{}_{\alpha k}\,\frac{\partial}{\partial x^h},\\
&\text{(c) } R^*\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^i} = R_i{}^h{}_{jk}\,\frac{\partial}{\partial x^h},
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
&\text{(a) } R^*\left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha} = R'\left(\frac{\delta}{\delta x^\gamma}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha} = R'_\alpha{}^\varepsilon{}_{\beta\gamma}\,\frac{\delta}{\delta x^\varepsilon},\\
&\text{(b) } R^*\left(\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha} = R'\left(\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\alpha} = R'_\alpha{}^\varepsilon{}_{\beta i}\,\frac{\delta}{\delta x^\varepsilon},\\
&\text{(c) } R^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)\frac{\delta}{\delta x^\alpha} = R'\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)\frac{\delta}{\delta x^\alpha} = R'_\alpha{}^\varepsilon{}_{ij}\,\frac{\delta}{\delta x^\varepsilon}.
\end{aligned} \tag{3.31}$$

Then by using (3.28), (3.30), (3.8), (3.21) and taking into account that the adapted tensor field $T_\alpha{}^i{}_\beta$ given by (2.19) is skew–symmetric with respect to lower indices we obtain:

$$R_i{}^h{}_{\alpha\beta} = \frac{\delta F_i{}^h{}_\alpha}{\delta x^\beta} - \frac{\delta F_i{}^h{}_\beta}{\delta x^\alpha} + F_i{}^j{}_\alpha F_j{}^h{}_\beta - F_i{}^j{}_\beta F_j{}^h{}_\alpha + C_i{}^h{}_j T_\alpha{}^j{}_\beta, \tag{3.32}$$

$$R_i{}^h{}_{\alpha k} = \frac{\partial F_i{}^h{}_\alpha}{\partial x^k} - \frac{\delta C_i{}^h{}_k}{\delta x^\alpha} + F_i{}^j{}_\alpha C_j{}^h{}_k - C_i{}^j{}_k F_j{}^h{}_\alpha + C_i{}^h{}_j\,\frac{\partial A^j_\alpha}{\partial x^k}, \tag{3.33}$$

$$R_i{}^h{}_{jk} = \frac{\partial C_i{}^h{}_j}{\partial x^k} - \frac{\partial C_i{}^h{}_k}{\partial x^j} + C_i{}^\ell{}_j C_\ell{}^h{}_k - C_i{}^\ell{}_k C_\ell{}^h{}_j. \tag{3.34}$$

Similarly, by using (3.29), (3.31), (3.7), (2.18) and (3.21) we deduce that:

$$R'_\alpha{}^\varepsilon{}_{\beta\gamma} = \frac{\delta F'_\alpha{}^\varepsilon{}_\beta}{\delta x^\gamma} - \frac{\delta F'_\alpha{}^\varepsilon{}_\gamma}{\delta x^\beta} + F'_\alpha{}^\mu{}_\beta F'_\mu{}^\varepsilon{}_\gamma - F'_\alpha{}^\mu{}_\gamma F'_\mu{}^\varepsilon{}_\beta + C'_\alpha{}^\varepsilon{}_j T_\beta{}^j{}_\gamma, \tag{3.35}$$

$$R'_\alpha{}^\varepsilon{}_{\beta i} = \frac{\partial F'_\alpha{}^\varepsilon{}_\beta}{\partial x^i} - \frac{\delta C'_\alpha{}^\varepsilon{}_i}{\delta x^\beta} + F'_\alpha{}^\mu{}_\beta C'_\mu{}^\varepsilon{}_i - C'_\alpha{}^\mu{}_i F'_\mu{}^\varepsilon{}_\beta + C'_\alpha{}^\varepsilon{}_j \frac{\partial A^j_\beta}{\partial x^i}, \tag{3.36}$$

$$R'_\alpha{}^\varepsilon{}_{ij} = \frac{\partial C'_\alpha{}^\varepsilon{}_i}{\partial x^j} - \frac{\partial C'_\alpha{}^\varepsilon{}_j}{\partial x^i} + C'_\alpha{}^\beta{}_i C'_\beta{}^\varepsilon{}_j - C'_\alpha{}^\beta{}_j C'_\beta{}^\varepsilon{}_i. \tag{3.37}$$

Remark 3.2. It is easy to check that (3.32)–(3.34) and (3.35)–(3.37) can be also obtained from (1.2.20) and (1.2.21) respectively by replacing the non–holonomic frame field $\{E_i, E_\alpha\}$ by the semi–holonomic frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$. ■

Taking into account (3.30) and (3.31), we state the following.

Theorem 3.8. *The local components of the curvature tensor field of the adapted linear connection* $\nabla^* = (\nabla, \nabla')$ *with respect to the semi–holonomic frame field* $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ *are given by* (3.32)–(3.37).

Next, we proceed with local components for torsion tensor fields of ∇^*, ∇ and ∇'. Denote by T^* the torsion tensor field of ∇^* and by using (1.2.14), (3.7), (3.8), (3.21) and (2.18) we obtain

$$\begin{aligned}
T^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) &= \left(C_i{}^k{}_j - C_j{}^k{}_i\right)\frac{\partial}{\partial x^k}, \\
T^*\left(\frac{\partial}{\partial x^j}, \frac{\delta}{\delta x^\alpha}\right) &= -T^*\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^j}\right) \\
&= \left(\frac{\partial A^k_\alpha}{\partial x^j} - F_j{}^k{}_\alpha\right)\frac{\partial}{\partial x^k} + C'_\alpha{}^\gamma{}_j \frac{\delta}{\delta x^\gamma}, \\
T^*\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) &= T_\alpha{}^k{}_\beta \frac{\partial}{\partial x^k} + \left(F'_\alpha{}^\gamma{}_\beta - F'_\beta{}^\gamma{}_\alpha\right)\frac{\delta}{\delta x^\gamma}.
\end{aligned} \tag{3.38}$$

On the other hand, we set:

$$\begin{aligned}
T^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) &= T^*{}_i{}^k{}_j \frac{\partial}{\partial x^k}, \\
T^*\left(\frac{\partial}{\partial x^j}, \frac{\delta}{\delta x^\alpha}\right) &= T^*{}_\alpha{}^k{}_j \frac{\partial}{\partial x^k} + T^*{}_\alpha{}^\gamma{}_j \frac{\delta}{\delta x^\gamma}, \\
T^*\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) &= T^*{}_\alpha{}^k{}_\beta \frac{\partial}{\partial x^k} + T^*{}_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma}.
\end{aligned} \tag{3.39}$$

Comparing (3.38) and (3.39) we obtain the following.

Theorem 3.9. *The local components of the torsion tensor field of the adapted linear connection* $\nabla^* = (\nabla, \nabla')$ *are given by*

$$\begin{aligned}
&\text{(a) } T^{*}{}_{i}{}^{k}{}_{j} = C_{i}{}^{k}{}_{j} - C_{j}{}^{k}{}_{i}, \text{ (b) } T^{*}{}_{\alpha}{}^{k}{}_{j} = \frac{\partial A^k_\alpha}{\partial x^j} - F_{j}{}^{k}{}_{\alpha},\\
&\text{(c) } T^{*}{}_{\alpha}{}^{\gamma}{}_{j} = C'{}_{\alpha}{}^{\gamma}{}_{j}, \qquad \text{(d) } T^{*}{}_{\alpha}{}^{\gamma}{}_{\beta} = F'{}_{\alpha}{}^{\gamma}{}_{\beta} - F'{}_{\beta}{}^{\gamma}{}_{\alpha},\\
&\text{(e) } T^{*}{}_{\alpha}{}^{k}{}_{\beta} = T_{\alpha}{}^{k}{}_{\beta} = \frac{\delta A^k_\alpha}{\delta x^\beta} - \frac{\delta A^k_\beta}{\delta x^\alpha}\,.
\end{aligned} \tag{3.40}$$

In Section 1.2, by using the Otsuki connections on a vector bundle we defined a torsion tensor field for a linear connection on a distribution. More precisely, according to (1.2.25) and (1.2.26), the linear connections ∇ and ∇' on $\mathcal{D}$ and $\mathcal{D}'$ have the torsion tensor fields:

$$T(X, QY) = \nabla_X QY - \nabla_{QY} QX - Q[X, QY], \tag{3.41}$$

and

$$T'(X, Q'Y) = \nabla'_X Q'Y - \nabla'_{Q'Y} Q'X - Q'[X, Q'Y], \tag{3.42}$$

respectively, for any $X, Y \in \Gamma(TM)$. Now, take the semi–holonomic frame field $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\alpha}\right\}$ and put:

$$\begin{aligned}
&\text{(a) } T\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = T_{i}{}^{k}{}_{j}\, \frac{\partial}{\partial x^k}\,,\\
&\text{(b) } T\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^i}\right) = -T_{\alpha}{}^{k}{}_{i}\, \frac{\partial}{\partial x^k}\,,
\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}
&\text{(a) } T'\left(\frac{\partial}{\partial x^j}, \frac{\delta}{\delta x^\alpha}\right) = T'{}_{\alpha}{}^{\gamma}{}_{j}\, \frac{\delta}{\delta x^\gamma}\,,\\
&\text{(b) } T'\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) = T'{}_{\alpha}{}^{\gamma}{}_{\beta}\, \frac{\delta}{\delta x^\gamma}\,.
\end{aligned} \tag{3.44}$$

Then, by using (3.41), (3.42), (3.7), (3.8) and (3.21) we obtain

$$\text{(a) } T_{i}{}^{k}{}_{j} = C_{i}{}^{k}{}_{j} - C_{j}{}^{k}{}_{i} = T^{*}{}_{i}{}^{k}{}_{j}, \quad \text{(b) } T_{\alpha}{}^{k}{}_{i} = \frac{\partial A^k_\alpha}{\partial x^i} - F_{i}{}^{k}{}_{\alpha} = T^{*}{}_{\alpha}{}^{k}{}_{i}, \tag{3.45}$$

and

$$\text{(a) } T'{}_{\alpha}{}^{\gamma}{}_{j} = C'{}_{\alpha}{}^{\gamma}{}_{j} = T^{*}{}_{\alpha}{}^{\gamma}{}_{j}, \quad \text{(b) } T'{}_{\alpha}{}^{\gamma}{}_{\beta} = F'{}_{\alpha}{}^{\gamma}{}_{\beta} - F'{}_{\beta}{}^{\gamma}{}_{\alpha} = T^{*}{}_{\alpha}{}^{\gamma}{}_{\beta}. \tag{3.46}$$

Therefore, we can state the following.

Theorem 3.10. *The local components of the torsion tensor fields of the structural and transversal linear connections* ∇ *and* ∇' *with respect to the semi–holonomic frame field* $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\alpha}\right\}$ *are given by* (3.45) *and* (3.46) *respectively.*

Remark 3.3. The local components of the curvature and torsion tensor fields of the structural and transversal linear connections ∇ and ∇' with respect to a semi–holonomic frame field define adapted tensor fields on $(M, \mathcal{F})$. ■

Remark 3.4. The Schouten–Van Kampen and Vrănceanu connections are examples of adapted connections on a foliated manifold. We shall make use of them in Chapter 3 for studying foliated manifolds endowed with a Riemannian (semi–Riemannian) metric. ■

2.4 Ricci and Bianchi Identities

Let $(M, \mathcal{F})$ be a foliated manifold with $\mathcal{D}$ the structural distribution and $\mathcal{D}'$ a transversal distribution on M. Suppose that ∇ and ∇' are structural and transversal connections on M. In the present section we use both the structural and transversal covariant derivatives in order to obtain Ricci and Bianchi identities for ∇ and ∇'.

First, we consider a structural vector field $U = U^i \dfrac{\partial}{\partial x^i}$ and by using (3.8), (3.13) and (3.14) obtain

$$\text{(a) } \nabla_{\frac{\delta}{\delta x^\alpha}} U = {U^i}_{|\alpha} \frac{\partial}{\partial x^i}, \quad \text{(b) } \nabla_{\frac{\partial}{\partial x^j}} U = {U^i}_{\|j} \frac{\partial}{\partial x^i}. \tag{4.1}$$

By direct calculations using transversal and structural covariant derivatives of adapted tensor fields (see (3.17) and (3.18)) we obtain the following covariant derivatives of order two:

$$\nabla_{\frac{\delta}{\delta x^\beta}} \nabla_{\frac{\delta}{\delta x^\alpha}} U = \left({U^i}_{|\alpha|\beta} + {U^i}_{|\gamma} {F'_\alpha}^\gamma{}_\beta\right) \frac{\partial}{\partial x^i}, \tag{4.2}$$

$$\nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\delta}{\delta x^\alpha}} U = \left({U^i}_{|\alpha\|j} + {U^i}_{|\gamma} {C'_\alpha}^\gamma{}_j\right) \frac{\partial}{\partial x^i}, \tag{4.3}$$

$$\nabla_{\frac{\delta}{\delta x^\alpha}} \nabla_{\frac{\partial}{\partial x^j}} U = \left({U^i}_{\|j|\alpha} + {U^i}_{\|h} {F_j}^h{}_\alpha\right) \frac{\partial}{\partial x^i}, \tag{4.4}$$

$$\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} U = \left({U^i}_{\|j\|k} + {U^i}_{\|h} {C_j}^h{}_k\right) \frac{\partial}{\partial x^i}. \tag{4.5}$$

Then by using (4.2)–(4.5) in (3.28) and taking into account (3.30), (2.18), (3.21), (3.45) and (3.46) we obtain the following identities:

$$U^i{}_{|\alpha|\beta} - U^i{}_{|\beta|\alpha} = U^j R_j{}^i{}_{\alpha\beta} - U^i{}_{|\gamma} T'{}_\alpha{}^\gamma{}_\beta - U^i{}_{\|k} T_\alpha{}^k{}_\beta, \tag{4.6}$$

$$U^i{}_{|\alpha\|j} - U^i{}_{\|j|\alpha} = U^h R_h{}^i{}_{\alpha j} - U^i{}_{|\gamma} C'{}_\alpha{}^\gamma{}_j - U^i{}_{\|k} T_\alpha{}^k{}_j, \tag{4.7}$$

$$U^i{}_{\|j\|k} - U^i{}_{\|k\|j} = U^h R_h{}^i{}_{jk} - U^i{}_{\|h} T_j{}^h{}_k. \tag{4.8}$$

Next, we consider a transversal vector field $Z = Z^\alpha \dfrac{\delta}{\delta x^\alpha}$, and by using (3.7), (3.15) and (3.16) we obtain

$$\text{(a) } \nabla'_{\frac{\delta}{\delta x^\beta}} Z = Z^\alpha{}_{|\beta} \frac{\delta}{\delta x^\alpha}, \quad \text{(b) } \nabla'_{\frac{\partial}{\partial x^i}} Z = Z^\alpha{}_{\|i} \frac{\delta}{\delta x^\alpha}. \tag{4.9}$$

Then we deduce that

$$\nabla'_{\frac{\delta}{\delta x^\gamma}} \nabla'_{\frac{\delta}{\delta x^\beta}} Z = \left(Z^\alpha{}_{|\beta|\gamma} + Z^\alpha{}_{|\varepsilon} F'{}_\beta{}^\varepsilon{}_\gamma\right) \frac{\delta}{\delta x^\alpha}, \tag{4.10}$$

$$\nabla'_{\frac{\partial}{\partial x^j}} \nabla'_{\frac{\delta}{\delta x^\beta}} Z = \left(Z^\alpha{}_{|\beta\|j} + Z^\alpha{}_{|\varepsilon} C'{}_\beta{}^\varepsilon{}_j\right) \frac{\delta}{\delta x^\alpha}, \tag{4.11}$$

$$\nabla'_{\frac{\delta}{\delta x^\beta}} \nabla'_{\frac{\partial}{\partial x^j}} Z = \left(Z^\alpha{}_{\|j|\beta} + Z^\alpha{}_{\|k} F_j{}^k{}_\beta\right) \frac{\delta}{\delta x^\alpha}, \tag{4.12}$$

$$\nabla'_{\frac{\partial}{\partial x^k}} \nabla'_{\frac{\partial}{\partial x^j}} Z = \left(Z^\alpha{}_{\|j\|k} + Z^\alpha{}_{\|h} C_j{}^h{}_k\right) \frac{\delta}{\delta x^\alpha}. \tag{4.13}$$

Finally, by using (4.10)–(4.13) in (3.29) and taking into account (3.31), (2.18), (3.21), (3.45) and (3.46) we obtain the identities:

$$Z^\alpha{}_{|\beta|\gamma} - Z^\alpha{}_{|\gamma|\beta} = Z^\varepsilon R'{}_\varepsilon{}^\alpha{}_{\beta\gamma} - Z^\alpha{}_{|\varepsilon} T'{}_\beta{}^\varepsilon{}_\gamma - Z^\alpha{}_{\|i} T_\beta{}^i{}_\gamma, \tag{4.14}$$

$$Z^\alpha{}_{|\beta\|j} - Z^\alpha{}_{\|j|\beta} = Z^\varepsilon R'{}_\varepsilon{}^\alpha{}_{\beta j} - Z^\alpha{}_{|\varepsilon} C'{}_\beta{}^\varepsilon{}_j - Z^\alpha{}_{\|i} T_\beta{}^i{}_j, \tag{4.15}$$

$$Z^\alpha{}_{\|j\|k} - Z^\alpha{}_{\|k\|j} = Z^\varepsilon R'{}_\varepsilon{}^\alpha{}_{jk} - Z^\alpha{}_{\|i} T_j{}^i{}_k. \tag{4.16}$$

According to the name given for such identities in case of a linear connection on a manifold, we call the groups of identities {(4.6), (4.7), (4.8)} and {(4.14), (4.15), (4.16)} the **structural Ricci identities** and **transversal Ricci identities** respectively on the foliated manifold $(M, \mathcal{F})$.

In order to obtain some Bianchi identities for both the structural and transversal linear connections ∇ and ∇' we consider the adapted linear connection $\nabla^* = (\nabla, \nabla')$ given by

$$\nabla^*_X Y = \nabla_X QY + \nabla'_X Q'Y, \quad \forall\, X, Y \in \Gamma(TM). \tag{4.17}$$

Then we recall the Bianchi identities (see Kobayashi–Nomizu [KN63], p. 135) for the linear connection ∇^*:

$$\sum_{(X,Y,Z)} \{\nabla^*_X T^*)(Y, Z) + T^*(T^*(X, Y), Z) - R^*(X, Y)Z\} = 0, \tag{4.18}$$

and

$$\sum_{(X,Y,Z)} \{(\nabla^*_X R^*)(Y,Z) + R^*(T^*(X,Y),Z)\}(U) = 0, \tag{4.19}$$

for any $X, Y, Z, U \in \Gamma(TM)$, where $\sum_{(X,Y,Z)}$ denotes the cyclic sum with respect to X, Y, Z and T^* and R^* are the torsion and curvature tensor fields of ∇^*. For local expressions of (4.18) we have to consider the following cases.

Case I. $X = \dfrac{\delta}{\delta x^\gamma}\,, Y = \dfrac{\delta}{\delta x^\beta}\,, Z = \dfrac{\delta}{\delta x^\alpha}\cdot$
Then by direct calculations using (3.39), (3.45), (3.46) and (3.17) we obtain

$$\left(\nabla^*_{\frac{\delta}{\delta x^\gamma}} T^*\right)\left(\frac{\delta}{\delta x^\beta}\,, \frac{\delta}{\delta x^\alpha}\right) = T_\alpha{}^i{}_{\beta|\gamma}\,\frac{\partial}{\partial x^i} + T'{}_\alpha{}^\varepsilon{}_{\beta|\gamma}\,\frac{\delta}{\delta x^\varepsilon}\,, \tag{4.20}$$

and

$$\begin{aligned} T^*\left(T^*\left(\frac{\delta}{\delta x^\gamma}\,, \frac{\delta}{\delta x^\beta}\right), \frac{\delta}{\delta x^\alpha}\right) &= \left(T_\alpha{}^i{}_j T_\beta{}^j{}_\gamma + T_\alpha{}^i{}_\varepsilon T'{}_\beta{}^\varepsilon{}_\gamma\right)\frac{\partial}{\partial x^i} \\ &\quad + \left(C'{}_\alpha{}^\varepsilon{}_j T_\beta{}^j{}_\gamma + T'{}_\alpha{}^\varepsilon{}_\mu T'{}_\beta{}^\mu{}_\gamma\right)\frac{\delta}{\delta x^\varepsilon}\cdot \end{aligned} \tag{4.21}$$

We now use (4.20), (4.21) and (3.31a) and taking into account that $\left\{\dfrac{\delta}{\delta x^\varepsilon}\right\}$ and $\left\{\dfrac{\partial}{\partial x^i}\right\}$ are local bases for $\Gamma(\mathcal{D}')$ and $\Gamma(\mathcal{D})$ respectively, we deduce the identities:

$$\sum_{(\alpha,\beta,\gamma)} \left\{T_\alpha{}^i{}_{\beta|\gamma} + T_\alpha{}^i{}_j T_\beta{}^j{}_\gamma + T_\alpha{}^i{}_\varepsilon T'{}_\beta{}^\varepsilon{}_\gamma\right\} = 0, \tag{4.22}$$

and

$$\sum_{(\alpha,\beta,\gamma)} \left\{T'{}_\alpha{}^\varepsilon{}_{\beta|\gamma} + C'{}_\alpha{}^\varepsilon{}_j T_\beta{}^j{}_\gamma + T'{}_\alpha{}^\varepsilon{}_\mu T'{}_\beta{}^\mu{}_\gamma - R'{}_\alpha{}^\varepsilon{}_{\beta\gamma}\right\} = 0, \tag{4.23}$$

where $\sum_{(\alpha,\beta,\gamma)}$ denotes the cyclic sum with respect to (α, β, γ).

Similarly, we obtain the local expressions of (4.18) for the next three cases.

Case II. $X = \dfrac{\partial}{\partial x^k}\,, Y = \dfrac{\delta}{\delta x^\beta}\,, Z = \dfrac{\delta}{\delta x^\alpha}\cdot$

$$\begin{aligned} &T_\alpha{}^i{}_{\beta\|k} + T_\beta{}^i{}_{k|\alpha} - T_\alpha{}^i{}_{k|\beta} + T_k{}^i{}_j T_\alpha{}^j{}_\beta - T'{}_\alpha{}^\varepsilon{}_\beta T_\varepsilon{}^i{}_k + T_\alpha{}^i{}_j T_\beta{}^j{}_k \\ &\quad - T_\beta{}^i{}_j T_\alpha{}^j{}_k + T_\alpha{}^i{}_\varepsilon C'{}_\beta{}^\varepsilon{}_k - T_\beta{}^i{}_\varepsilon C'{}_\alpha{}^\varepsilon{}_k - R_k{}^i{}_{\alpha\beta} = 0, \end{aligned} \tag{4.24}$$

$$\begin{aligned} &T'_{\alpha}{}^{\gamma}{}_{\beta\|k} + C'_{\beta}{}^{\gamma}{}_{k|\alpha} - C'_{\alpha}{}^{\gamma}{}_{k|\beta} - T'_{\alpha}{}^{\varepsilon}{}_{\beta} C'_{\varepsilon}{}^{\gamma}{}_{k} + C'_{\alpha}{}^{\gamma}{}_{j} T_{\beta}{}^{j}{}_{k} \\ &\qquad - C'_{\beta}{}^{\gamma}{}_{j} T_{\alpha}{}^{j}{}_{k} + T'_{\alpha}{}^{\gamma}{}_{\varepsilon} C'_{\beta}{}^{\varepsilon}{}_{k} - T'_{\beta}{}^{\gamma}{}_{\varepsilon} C'_{\alpha}{}^{\varepsilon}{}_{k} \\ &\qquad + R'_{\beta}{}^{\gamma}{}_{\alpha k} - R'_{\alpha}{}^{\gamma}{}_{\beta k} = 0. \end{aligned} \tag{4.25}$$

Case III. $X = \dfrac{\partial}{\partial x^k}\,,\ Y = \dfrac{\partial}{\partial x^j}\,,\ Z = \dfrac{\delta}{\delta x^\alpha}\,\cdot$

$$\begin{aligned} &T_{\alpha}{}^{i}{}_{j\|k} - T_{\alpha}{}^{i}{}_{k\|j} + T_{j}{}^{i}{}_{k|\alpha} + T_{\alpha}{}^{i}{}_{h} T_{j}{}^{h}{}_{k} + T_{k}{}^{i}{}_{h} T_{\alpha}{}^{h}{}_{j} - T_{j}{}^{i}{}_{h} T_{\alpha}{}^{h}{}_{k} \\ &\qquad + C'_{\alpha}{}^{\varepsilon}{}_{k} T_{\varepsilon}{}^{i}{}_{j} - C'_{\alpha}{}^{\varepsilon}{}_{j} T_{\varepsilon}{}^{i}{}_{k} + R_{j}{}^{i}{}_{\alpha k} - R_{k}{}^{i}{}_{\alpha j} = 0, \end{aligned} \tag{4.26}$$

$$\begin{aligned} &C'_{\alpha}{}^{\gamma}{}_{j\|k} - C'_{\alpha}{}^{\gamma}{}_{k\|j} \\ &+ C'_{\alpha}{}^{\gamma}{}_{h} T_{j}{}^{h}{}_{k} + C'_{\alpha}{}^{\varepsilon}{}_{k} C'_{\varepsilon}{}^{\gamma}{}_{j} - C'_{\alpha}{}^{\varepsilon}{}_{j} C'_{\varepsilon}{}^{\gamma}{}_{k} - R'_{\alpha}{}^{\varepsilon}{}_{jk} = 0. \end{aligned} \tag{4.27}$$

Case IV. $X = \dfrac{\partial}{\partial x^k}\,,\ Y = \dfrac{\partial}{\partial x^j}\,,\ Z = \dfrac{\partial}{\partial x^i}\,\cdot$

$$\sum_{(i,j,k)} \left\{ T_{i}{}^{h}{}_{j\|k} + T_{i}{}^{h}{}_{r} T_{j}{}^{r}{}_{k} - R_{i}{}^{h}{}_{jk} \right\} = 0. \tag{4.28}$$

The local expressions for (4.19) are obtained by considering eight cases.

Case I. $X = \dfrac{\delta}{\delta x^\gamma}\,,\ Y = \dfrac{\delta}{\delta x^\beta}\,,\ Z = \dfrac{\delta}{\delta x^\alpha}\,,\ U = \dfrac{\partial}{\partial x^i}\,\cdot$

$$\sum_{(\alpha,\beta,\gamma)} \left\{ R_{i}{}^{h}{}_{\alpha\beta|\gamma} + R_{i}{}^{h}{}_{\alpha j} T_{\beta}{}^{j}{}_{\gamma} + R_{i}{}^{h}{}_{\alpha\varepsilon} T'_{\beta}{}^{\varepsilon}{}_{\gamma} \right\} = 0. \tag{4.29}$$

Case II. $X = \dfrac{\delta}{\delta x^\gamma}\,,\ Y = \dfrac{\delta}{\delta x^\beta}\,,\ Z = \dfrac{\partial}{\partial x^j}\,,\ U = \dfrac{\partial}{\partial x^i}\,\cdot$

$$\begin{aligned} &R_{i}{}^{h}{}_{\beta\gamma\|j} + R_{i}{}^{h}{}_{\gamma j|\beta} - R_{i}{}^{h}{}_{\beta j|\gamma} + R_{i}{}^{h}{}_{jk} T_{\beta}{}^{k}{}_{\gamma} - R_{i}{}^{h}{}_{\varepsilon j} T'_{\beta}{}^{\varepsilon}{}_{\gamma} \\ &\qquad + R_{i}{}^{h}{}_{\beta k} T_{\gamma}{}^{k}{}_{j} - R_{i}{}^{h}{}_{\gamma k} T_{\beta}{}^{k}{}_{j} + R_{i}{}^{h}{}_{\beta\varepsilon} C'_{\gamma}{}^{\varepsilon}{}_{j} \\ &\qquad - R_{i}{}^{h}{}_{\gamma\varepsilon} C'_{\beta}{}^{\varepsilon}{}_{j} = 0. \end{aligned} \tag{4.30}$$

Case III. $X = \dfrac{\delta}{\delta x^\gamma}\,,\ Y = \dfrac{\partial}{\partial x^k}\,,\ Z = \dfrac{\partial}{\partial x^j}\,,\ U = \dfrac{\partial}{\partial x^i}\,\cdot$

$$\begin{aligned} &R_{i}{}^{h}{}_{jk|\gamma} + R_{i}{}^{h}{}_{\gamma j\|k} - R_{i}{}^{h}{}_{\gamma k\|j} + R_{i}{}^{h}{}_{\gamma r} T_{j}{}^{r}{}_{k} + R_{i}{}^{h}{}_{kr} T_{\gamma}{}^{r}{}_{j} \\ &\qquad - R_{i}{}^{h}{}_{jr} T_{\gamma}{}^{r}{}_{k} + R_{i}{}^{h}{}_{\varepsilon j} C'_{\gamma}{}^{\varepsilon}{}_{k} - R_{i}{}^{h}{}_{\varepsilon k} C'_{\gamma}{}^{\varepsilon}{}_{j} = 0. \end{aligned} \tag{4.31}$$

Case IV. $X = \dfrac{\partial}{\partial x^h}\,,\; Y = \dfrac{\partial}{\partial x^k}\,,\; Z = \dfrac{\partial}{\partial x^j}\,,\; U = \dfrac{\partial}{\partial x^i}\,\cdot$

$$\sum_{(j,k,h)} \left\{ R_i{}^r{}_{jk\|h} + R_i{}^r{}_{js} T_k{}^s{}_h \right\} = 0. \tag{4.32}$$

Case V. $X = \dfrac{\delta}{\delta x^\gamma}\,,\; Y = \dfrac{\delta}{\delta x^\beta}\,,\; Z = \dfrac{\delta}{\delta x^\alpha}\,,\; U = \dfrac{\delta}{\delta x^\mu}\,\cdot$

$$\sum_{(\alpha,\beta,\gamma)} \left\{ R'_\mu{}^\varepsilon{}_{\alpha\beta|\gamma} + R'_\mu{}^\varepsilon{}_{\gamma i} T_\alpha{}^i{}_\beta + R'_\mu{}^\varepsilon{}_{\gamma\nu} T'_\alpha{}^\nu{}_\beta \right\} = 0. \tag{4.33}$$

Case VI. $X = \dfrac{\delta}{\delta x^\gamma}\,,\; Y = \dfrac{\delta}{\delta x^\beta}\,,\; Z = \dfrac{\partial}{\partial x^j}\,,\; U = \dfrac{\delta}{\delta x^\mu}\,\cdot$

$$\begin{aligned} R'_\mu{}^\nu{}_{\beta\gamma\|j} &+ R'_\mu{}^\nu{}_{\gamma j|\beta} - R'_\mu{}^\nu{}_{\beta j|\gamma} + R'_\mu{}^\nu{}_{jk} T_\beta{}^k{}_\gamma \\ &- R'_\mu{}^\nu{}_{\varepsilon j} T'_\beta{}^\varepsilon{}_\gamma + R'_\mu{}^\nu{}_{\beta k} T_\gamma{}^k{}_j - R'_\mu{}^\nu{}_{\gamma k} T_\beta{}^k{}_j \\ &+ R'_\mu{}^\nu{}_{\beta\varepsilon} C'_\gamma{}^\varepsilon{}_j - R'_\mu{}^\nu{}_{\gamma\varepsilon} C'_\beta{}^\varepsilon{}_j = 0. \end{aligned} \tag{4.34}$$

Case VII. $X = \dfrac{\delta}{\delta x^\gamma}\,,\; Y = \dfrac{\partial}{\partial x^k}\,,\; Z = \dfrac{\partial}{\partial x^j}\,,\; U = \dfrac{\delta}{\delta x^\mu}\,\cdot$

$$\begin{aligned} R'_\mu{}^\nu{}_{jk|\gamma} &+ R'_\mu{}^\nu{}_{\gamma j\|k} - R'_\mu{}^\nu{}_{\gamma k\|j} + R'_\mu{}^\nu{}_{\gamma h} T_j{}^h{}_k + R'_\mu{}^\nu{}_{kh} T_\gamma{}^h{}_j \\ &- R'_\mu{}^\nu{}_{jh} T_\gamma{}^h{}_k + R'_\mu{}^\nu{}_{\varepsilon j} C'_\gamma{}^\varepsilon{}_k - R'_\mu{}^\nu{}_{\varepsilon k} C'_\gamma{}^\varepsilon{}_j = 0. \end{aligned} \tag{4.35}$$

Case VIII. $X = \dfrac{\partial}{\partial x^h}\,,\; Y = \dfrac{\partial}{\partial x^k}\,,\; Z = \dfrac{\partial}{\partial x^j}\,,\; U = \dfrac{\delta}{\delta x^\mu}\,\cdot$

$$\sum_{(j,k,h)} \left\{ R'_\mu{}^\nu{}_{jk\|h} + R'_\mu{}^\nu{}_{jr} T_k{}^r{}_h \right\} = 0. \tag{4.36}$$

We call {(4.22), (4.24), (4.26), (4.28), (4.29)–(4.32)} and {(4.23), (4.25), (4.27), (4.33)–(4.36)} the **structural Bianchi identities** and **transversal Bianchi identities** respectively, corresponding to the adapted linear connection $\nabla^* = (\nabla, \nabla')$ on $(M, \mathcal{F})$.

Remark 4.1. The above Ricci and Bianchi identities were obtained by Bejancu–Farran [BF03a]. If in particular, we consider the foliation determined by the vertical bundle on the tangent bundle of a Finsler manifold, then we obtain all the Ricci and Bianchi identities for a Finsler connection (see Matsumoto [Mat86], pp.79,80, Bejancu–Farran [BF00a], pp.34,35). ∎

3

FOLIATIONS ON SEMI–RIEMANNIAN MANIFOLDS

In this chapter, we apply the results obtained in Chapter 1 and Chapter 2 to the elegant situation of a semi–Riemannian manifold with a non–degenerate foliation. In this case, there is a canonical distribution that is transversal to the foliation. In Sections 3.1 and 3.2 we study the Vrănceanu connection and the Schouten–Van Kampen connection and relate their geometry to the geometry of the foliation, the integrability of the transversal distribution, and to the geometry of the ambient manifold.

This approach enables us to extend the notion of foliations with bundle–like metrics to semi–Riemannian manifolds and to study their geometry. This is done in Section 3.3.

Section 3.4 is devoted to foliations with certain geometric features. Here we study foliations that are totally geodesic, totally umbilical, or minimal.

In the last section we discuss degenerate foliations of codimension one. This will be the first step towards degenerate foliations (of arbitrary codimension) that will be considered in the next chapter.

3.1 The Vrănceanu Connection on a Foliated Semi–Riemannian Manifold

Let (M, g) be an $(n+p)$–dimensional semi–Riemannian manifold and $\mathcal{F}$ be an n–foliation on M. We assume that the tangent distribution $\mathcal{D}$ to the foliation is semi–Riemannian, that is, the induced metric tensor field on $\mathcal{D}$ is non–degenerate and of constant index on M (see Section 1.4). Then we call $\mathcal{F}$ a **non–degenerate foliation** on (M, g), and $(M, g, \mathcal{F})$ is a **foliated semi–Riemannian manifold**. The complementary orthogonal distribution $\mathcal{D}^{\perp}$ to $\mathcal{D}$ in TM is semi–Riemannian too, and we take it as the **transversal distribution** to the foliation $\mathcal{F}$. Also, we call $\mathcal{D}$ the **structural distribution** of $\mathcal{F}$. The projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to the decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}, \tag{1.1}$$

are denoted by Q and Q' respectively. Then according to Theorem 1.5.1. we can state the following.

Theorem 1.1. *Let $\mathcal{D}$ and $\mathcal{D}^{\perp}$ be the structural and transversal distributions on the foliated semi–Riemannian manifold $(M, g, \mathcal{F})$. Then we have the following assertions:*

(i) *There exists a unique linear connection D on $\mathcal{D}$ satisfying the conditions:*

$$D_X QY - D_{QY} QX - Q[X, QY] = 0, \ \forall X, Y \in \Gamma(TM), \tag{1.2}$$

and

$$\begin{aligned}(D_{QX} g)(QY, QZ) = & \ QX(g(QY, QZ)) - g(D_{QX} QY, QZ) \\ & - g(QY, D_{QX} QZ) = 0, \quad \forall X, Y, Z \in \Gamma(TM).\end{aligned} \tag{1.3}$$

(ii) *There exists a unique linear connection $D^{\perp}$ on $\mathcal{D}^{\perp}$ satisfying the conditions:*

$$D^{\perp}_X Q'Y - D^{\perp}_{Q'Y} Q'X - Q'[X, Q'Y] = 0, \ \forall X, Y \in \Gamma(TM), \tag{1.4}$$

and

$$\begin{aligned}(D^{\perp}_{Q'X} g)(Q'Y, Q'Z) = & \ Q'X(g(Q'Y, Q'Z)) - g(D^{\perp}_{Q'X} Q'Y, Q'Z) \\ & - g(Q'Y, D^{\perp}_{Q'X} Q'Z) = 0, \ \forall X, Y, Z \in \Gamma(TM).\end{aligned} \tag{1.5}$$

Moreover, from (1.5.3) and (1.5.4) we see that D is given by

$$\begin{aligned}2g(D_{QX} QY, QZ) = & \ QX(g(QY, QZ)) + QY(g(QZ, QX)) \\ & - QZ(g(QX, QY)) + g([QX, QY], QZ) \\ & - g([QY, QZ], QX) + g([QZ, QX], QY),\end{aligned} \tag{1.6}$$

and

$$D_{Q'X} QY = Q[Q'X, QY], \tag{1.7}$$

for any $X, Y, Z \in \Gamma(TM)$. Similarly, we deduce that $D^{\perp}$ is given by

$$\begin{aligned}2g(D^{\perp}_{Q'X} Q'Y, Q'Z) = & \ Q'X(g(Q'Y, Q'Z)) + Q'Y(g(Q'Z, Q'X)) \\ & - Q'Z(g(Q'X, Q'Y)) + g([Q'X, Q'Y], Q'Z) \\ & - g([Q'Y, Q'Z], Q'X) + g([Q'Z, Q'X], Q'Y),\end{aligned} \tag{1.8}$$

and

$$D^{\perp}_{QX} Q'Y = Q'[QX, Q'Y], \tag{1.9}$$

for any $X, Y, Z \in \Gamma(TM)$. We keep for D and $D^{\perp}$ the names as **intrinsic linear connections** on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively (see Section 1.5).

It is common in the literature to use the quotient bundle $\widetilde{\mathcal{D}} = TM/\mathcal{D}$ when studying the geometry of the foliation $\mathcal{F}$. However, when M is a semi–Riemannian manifold, then $\widetilde{\mathcal{D}}$ is metric isomorphic to $\mathcal{D}^{\perp}$. Indeed, if $v \in T_xM$ defines the equivalence class $[v] \in T_xM/\mathcal{D}_x$, then $k_x : T_xM/\mathcal{D}_x \longrightarrow \mathcal{D}_x^{\perp}$; $k_x([v]) = Q'(v)$ defines a metric isomorphism $k : (\widetilde{\mathcal{D}}, k^*g) \longrightarrow (D^{\perp}, g)$, where k^*g is the pull–back of g by k. By using this isomorphism it is easily seen that our differential operator $D^{\perp} : \Gamma(\mathcal{D})\times\Gamma(\mathcal{D}^{\perp}) \longrightarrow \Gamma(\mathcal{D}^{\perp})$ defined by (1.9) gives what is known in the literature by **Bott connection** on $\mathcal{D}^{\perp} \approx \widetilde{\mathcal{D}}$. Though the Bott connection defines only structural covariant derivatives of transversal vector fields, it has all the properties of a usual linear connection on $\mathcal{D}^{\perp}$. This is a consequence of the fact that the Bott connection is the restriction of our intrinsic connection $D^{\perp}$ to $\Gamma(\mathcal{D})\times\Gamma(\mathcal{D}^{\perp})$.

Next, we consider the Levi–Civita connection $\widetilde{\nabla}$ on (M, g). Then comparing both (1.6) and (1.8) with (1.5.10) and taking into account (1.7) and (1.9) we obtain

$$\begin{aligned} &\text{(a) } D_X QY = Q\widetilde{\nabla}_{QX}QY + Q[Q'X, QY], \\ &\text{(b) } D^{\perp}_X Q'Y = Q'\widetilde{\nabla}_{Q'X}Q'Y + Q'[QX, Q'Y], \end{aligned} \tag{1.10}$$

for any $X, Y \in \Gamma(TM)$. By using (1.10b) and the above isomorphism we deduce that the intrinsic connection $D^{\perp}$ on $\mathcal{D}^{\perp}$ is just the linear connection ∇ defined by the formula (3.3) in Tondeur [Ton97], p.21, which has been used throughout that book and in several other works on foliations.

The adapted linear connection on $(M, g, \mathcal{F})$ determined by the pair $(D, D^{\perp})$ is the Vrănceanu connection ∇^* defined by the Levi–Civita connection $\widetilde{\nabla}$ (cf. Theorem 1.5.3).

By using some general formulas for adapted linear connections (see (1.2.4) and (1.3.16)) we deduce that the Vrănceanu connection is given either by

$$\nabla^*_X Y = D_X QY + D^{\perp}_X Q'Y, \tag{1.11}$$

or by

$$\nabla^*_X Y = Q\widetilde{\nabla}_{QX}QY + Q'\widetilde{\nabla}_{Q'X}Q'Y + Q[Q'X, QY] + Q'[QX, Q'Y], \tag{1.12}$$

for any $X, Y \in \Gamma(TM)$. Moreover, from Corollary 1.5.4 we see that the Vrănceanu connection ∇^* on $(M, g, \mathcal{F})$ is the only adapted linear connection on $(M, \mathcal{D}, \mathcal{D}^{\perp})$ satisfying the conditions:

$$\text{(a) } (\nabla^*_{QX} g)(QY, QZ) = 0, \quad \text{(b) } (\nabla^*_{Q'X} g)(Q'Y, Q'Z) = 0, \tag{1.13}$$

and

$$\text{(a) } Q(T^*(X, QY)) = 0, \quad \text{(b) } Q'(T^*(X, Q'Y)) = 0, \tag{1.14}$$

for any $X, Y \in \Gamma(TM)$, where T^* is the torsion tensor field of ∇^* given by

$$T^*(X, Y) = \nabla^*_X Y - \nabla^*_Y X - [X, Y]. \tag{1.15}$$

Finally, we note that the semi–Riemannian metric g is not parallel with respect to any of the intrinsic connections. More precisely, using (1.7), (1.9) and (1.11) we deduce that

$$\begin{aligned}(D_{Q'X}g)(QY,QZ) = \ & (\nabla^*_{Q'X}g)(QY,QZ) = Q'X(g(QY,QZ)) \\ & - g([Q'X,QY],QZ) - g([Q'X,QZ],QY),\end{aligned} \tag{1.16}$$

and

$$\begin{aligned}(D^{\perp}_{QX}g)(Q'Y,Q'Z) = \ & (\nabla^*_{QX}g)(Q'Y,Q'Z) = QX(g(Q'Y,Q'Z)) \\ & - g([QX,Q'Y],Q'Z) - g([QX,Q'Z],Q'Y),\end{aligned} \tag{1.17}$$

for any $X,Y,Z \in \Gamma(TM)$.

Now, we want to develop a study of the Vrănceanu connection in local coordinate systems. First, we consider the natural frame field $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^\alpha}\right\}$, where $\dfrac{\partial}{\partial x^i} \in \Gamma(\mathcal{D})$, $i \in \{1,...,n\}$, and put

$$\text{(a) } g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad \text{(b) } g_{i\alpha} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\right). \tag{1.18}$$

Taking into account that $\left\{\dfrac{\delta}{\delta x^\alpha}\right\}$, $\alpha \in \{n+1,...,n+p\}$, given by (2.2.3) are now orthogonal to $\left\{\dfrac{\partial}{\partial x^i}\right\}$, $i \in \{1,...,n\}$, we obtain

$$g_{j\alpha} - A^i_\alpha g_{ij} = 0. \tag{1.19}$$

Since $[g_{ij}]$ is the matrix of local components of the semi–Riemannian metric induced by g on $\mathcal{D}$, it has an inverse which we denote by $[g^{hk}]$. Then from (1.19) we deduce that

$$A^i_\alpha = g^{ij} g_{j\alpha}. \tag{1.20}$$

Next, we consider the semi–holonomic frame field $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\alpha}\right\}$, where

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - A^i_\alpha \frac{\partial}{\partial x^i}, \tag{1.21}$$

and A^i_α are given by (1.20). With respect to this frame field we set:

$$\begin{aligned}&\text{(a) } \nabla^*_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = C_i{}^k{}_j \frac{\partial}{\partial x^k}, \\ &\text{(b) } \nabla^*_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = D_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = D_i{}^k{}_\alpha \frac{\partial}{\partial x^k},\end{aligned} \tag{1.22}$$

and

$$
\begin{aligned}
&\text{(a) } \nabla^*_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = D^\perp_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = L_\alpha{}^\gamma{}_i \frac{\delta}{\delta x^\gamma}\,, \\
&\text{(b) } \nabla^*_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = D^\perp_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F_\alpha{}^\gamma{}_\beta \frac{\delta}{\delta x^\gamma}\,.
\end{aligned}
\tag{1.23}
$$

Also we put

$$
g_{\alpha\beta} = g\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right), \tag{1.24}
$$

and denote by $[g^{\gamma\mu}]$ the inverse matrix of $[g_{\alpha\beta}]$.

Proposition 1.2. *The local coefficients of the intrinsic connections D and $D^\perp$ with respect to the semi–holonomic frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ are given by*

$$
\text{(a) } C_i{}^k{}_j = \frac{1}{2} g^{kh}\left(\frac{\partial g_{hi}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right), \quad \text{(b) } D_i{}^k{}_\alpha = \frac{\partial A^k_\alpha}{\partial x^i}\,, \tag{1.25}
$$

and

$$
\text{(a) } L_\alpha{}^\beta{}_i = 0, \quad \text{(b) } F_\alpha{}^\beta{}_\gamma = \frac{1}{2} g^{\beta\mu}\left(\frac{\delta g_{\mu\alpha}}{\delta x^\gamma} + \frac{\delta g_{\mu\gamma}}{\delta x^\alpha} - \frac{\delta g_{\alpha\gamma}}{\delta x^\mu}\right), \tag{1.26}
$$

respectively.

Proof. By direct calculations using (1.6)–(1.9), (1.11), (1.18a), (1.22)–(1.24), (2.2.18) and (2.3.21). ■

Corollary 1.3. *The local coefficients of the Vrănceanu connection with respect to the semi–holonomic frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ are given by* (1.25) *and* (1.26).

Remark 1.1. By using the Cartan method of differential forms, Vaisman [Vai71] obtained the local coefficients given by (1.25) and (1.26) on a foliated Riemannian manifold $(M, g, \mathcal{F})$. He named the linear connection given locally by (1.25) and (1.26) the **second connection** on $(M, g, \mathcal{F})$, keeping the name **first connection** for Levi–Civita connection on (M, g). On the other hand, by using (1.12) we can easily see that the **adapted connection** $\nabla^{\mathcal{F}}$ defined by Reinhart [Rei83], p. 147 is just the Vrănceanu connection ∇^* on $(M, g, \mathcal{F})$. Taking into account that Vrănceanu [VG31] constructed first this connection on non–holonomic manifolds (see Sections 1.3 and 1.5), throughout the book, we call ∇^* given invariantly by (1.2) and locally by (1.25) and (1.26), the **Vrănceanu connection** on $(M, g, \mathcal{F})$. ■

Now, we deduce the local components of the torsion and curvature tensor fields of ∇^*. First, we prove the following.

Proposition 1.4. *The local components of the torsion tensor field* T^* *of the Vrănceanu connection are given by*

$$\begin{aligned}&\text{(a) } T^*{}_i{}^k{}_j = 0, \quad \text{(b) } T^*{}_\alpha{}^k{}_i = 0, \quad \text{(c) } T^*{}_\alpha{}^\gamma{}_i = 0,\\ &\text{(d) } T^*{}_\alpha{}^\gamma{}_\beta = 0, \text{ (e) } T^*{}_\alpha{}^k{}_\beta = \frac{\delta A^k_\alpha}{\delta x^\beta} - \frac{\delta A^k_\beta}{\delta x^\alpha}\,,\end{aligned} \tag{1.27}$$

where A^k_α *are given by* (1.20).

Proof. By using (1.25) and (1.26) into (2.3.40). ■

As $T^*{}_\alpha{}^k{}_\beta$ is the integrability tensor for the transversal distribution (see (2.2.18)–(2.2.20)), we can state the following.

Theorem 1.5. *The transversal distribution to the foliation* $\mathcal{F}$ *is integrable if and only if the Vrănceanu connection on* $(M, g, \mathcal{F})$ *is torsion–free.*

Finally, by using (1.22) and (1.23) in (2.3.32)–(2.3.37) we obtain the following.

Proposition 1.6. *The local components of the curvature tensor fields* R *and* R' *of the intrinsic connections* D *and* $D^\perp$ *with respect to the semi–holonomic frame field* $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ *are given by*

$$\begin{aligned}R_i{}^h{}_{\alpha\beta} = {} & \frac{\delta D_i{}^h{}_\alpha}{\delta x^\beta} - \frac{\delta D_i{}^h{}_\beta}{\delta x^\alpha} + D_i{}^j{}_\alpha \; D_j{}^h{}_\beta\\ & - D_i{}^j{}_\beta \; D_j{}^h{}_\alpha + C_i{}^h{}_j \; T^*{}_\alpha{}^j{}_\beta,\end{aligned} \tag{1.28}$$

$$\begin{aligned}R_i{}^h{}_{\alpha k} = {} & \frac{\partial D_i{}^h{}_\alpha}{\partial x^k} - \frac{\delta C_i{}^h{}_k}{\delta x^\alpha} + D_i{}^j{}_\alpha \; C_j{}^h{}_k\\ & - C_i{}^j{}_\kappa \; D_j{}^h{}_\alpha + C_i{}^h{}_j \; D_k{}^j{}_\alpha,\end{aligned} \tag{1.29}$$

$$R_i{}^h{}_{jk} = \frac{\partial C_i{}^h{}_j}{\partial x^k} - \frac{\partial C_i{}^h{}_k}{\partial x^j} + C_i{}^\ell{}_j \; C_\ell{}^h{}_k - C_i{}^\ell{}_k \; C_\ell{}^h{}_j, \tag{1.30}$$

$$R'{}_\alpha{}^\varepsilon{}_{\beta\gamma} = \frac{\delta F_\alpha{}^\varepsilon{}_\beta}{\delta x^\gamma} - \frac{\delta F_\alpha{}^\varepsilon{}_\gamma}{\delta x^\beta} + F_\alpha{}^\mu{}_\beta \; F_\mu{}^\varepsilon{}_\gamma - F_\alpha{}^\mu{}_\gamma \; F_\mu{}^\varepsilon{}_\beta, \tag{1.31}$$

$$R'{}_\alpha{}^\varepsilon{}_{\beta i} = \frac{\partial F_\alpha{}^\varepsilon{}_\beta}{\partial x^i}\,, \tag{1.32}$$

$$R'{}_\alpha{}^\varepsilon{}_{ij} = 0, \tag{1.33}$$

where in the left hand side we use the notations from (2.3.30) *and* (2.3.31).

Taking into account that $\nabla^* = (D, D^\perp)$ we deduce that all the local components of the curvature tensor field of the Vrănceanu connection are given by (1.28)–(1.33).

By using the local coefficients of the Vrănceanu connection ∇^* on $(M, g, \mathcal{F})$ we can define the **transversal** and **structural Vrănceanu covariant derivatives** of an adapted tensor field $T = \left(T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}\right)$ as follows (see (2.3.17) and (2.3.18)):

$$\begin{aligned} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t|\gamma} &= \frac{\delta T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}}{\delta x^\gamma} \\ &+\sum_{a=1}^{q} T^{i_1\dots h i_{a+1}\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t} \, D_h{}^{i_a}{}_\gamma + \sum_{b=1}^{r} T^{i_1\dots i_q\alpha_1\dots\varepsilon\alpha_{b+1}\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t} \, F_\varepsilon{}^{\alpha_b}{}_\gamma \\ &-\sum_{c=1}^{s} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots h j_{c+1}\dots j_s\beta_1\dots\beta_t} \, D_{j_c}{}^h{}_\gamma - \sum_{d=1}^{t} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\varepsilon\beta_{d+1}\dots\beta_t} \, F_{\beta_d}{}^\varepsilon{}_\gamma, \end{aligned} \tag{1.34}$$

and

$$\begin{aligned} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t\|k} &= \frac{\partial T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t}}{\partial x^k} \\ &+\sum_{a=1}^{q} T^{i_1\dots h i_{a+1}\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots j_s\beta_1\dots\beta_t} \, C_h{}^{i_a}{}_k - \sum_{c=1}^{s} T^{i_1\dots i_q\alpha_1\dots\alpha_r}_{j_1\dots h j_{c+1}\dots j_s\beta_1\dots\beta_t} \, C_{j_c}{}^h{}_k, \end{aligned} \tag{1.35}$$

respectively.

When the transversal (resp. structural) Vrănceanu covariant derivative of T vanishes identically on M, we say that T is **transversal** (resp. **structural**) **Vrănceanu parallel**. As examples we have the adapted tensor fields g_{ij} and $g_{\alpha\beta}$ which are structural and transversal Vrănceanu parallel respectively (see Proposition 1.8).

Remark 1.2. Each of these covariant derivatives is defined by using both intrinsic connections D and $D^\perp$. If we consider only the connection $D^\perp = (L_\alpha{}^\gamma{}_i, F_\alpha{}^\gamma{}_\beta)$ on the transversal distribution (which was considered so far in the literature), none of the above covariant derivatives can be defined. Thus from this point of view, our study is completely different from what is known in the literature. ■

Due to (1.26a) the structural Vrănceanu covariant derivative has a substantially simplified form in comparison with (2.3.18). In particular, for a transversal tensor field $T = \left(T^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_t}\right)$, from (1.35) we deduce that

$$T^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_t\|k} = \frac{\partial T^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_t}}{\partial x^k}\,. \tag{1.36}$$

Then taking into account the definition of basic transversal tensor fields (see (2.2.14)) and (1.36) we obtain the following.

Theorem 1.7. *A transversal tensor field on a foliated semi–Riemannian manifold $(M, g, \mathcal{F})$ is basic if and only if it is structural Vrănceanu parallel.*

We now exemplify the above covariant derivatives for three classes of adapted tensor fields: vector fields, 1–forms and semi–Riemannian metrics. First, if $X = X^i\partial/\partial x^i$ and $Y = Y^\alpha\delta/\delta x^\alpha$ are structural and transversal vector fields, then we have

$$\text{(a) } X^i{}_{|\gamma} = \frac{\delta X^i}{\delta x^\gamma} + X^j D_j{}^i{}_\gamma, \quad \text{(b) } X^i{}_{\|k} = \frac{\partial X^i}{\partial x^k} + X^j C_j{}^i{}_k, \tag{1.37}$$

and

$$\text{(a) } Y^\alpha{}_{|\gamma} = \frac{\delta Y^\alpha}{\delta x^\gamma} + Y^\beta F_\beta{}^\alpha{}_\gamma, \quad \text{(b) } Y^\alpha{}_{\|k} = \frac{\partial Y^\alpha}{\partial x^k}, \tag{1.38}$$

respectively. Similarly, for $\omega = \omega_i\delta x^i$ and $\theta = \theta_\alpha dx^\alpha$, we obtain:

$$\text{(a) } \omega_{i|\gamma} = \frac{\delta\omega_i}{\delta x^\gamma} - \omega_j D_i{}^j{}_\gamma, \quad \text{(b) } \omega_{i\|k} = \frac{\partial\omega_i}{\partial x^k} - \omega_j C_i{}^j{}_k, \tag{1.39}$$

and

$$\text{(a) } \theta_{\alpha|\gamma} = \frac{\delta\theta_\alpha}{\delta x^\gamma} - \theta_\beta F_\alpha{}^\beta{}_\gamma, \quad \text{(b) } \theta_{\alpha\|k} = \frac{\partial\theta_\alpha}{\partial x^k}, \tag{1.40}$$

respectively. Finally, we note that the semi–Riemannian metric on $\mathcal{D}$ (resp. $\mathcal{D}^\perp$) is a structural (resp. transversal) tensor field with local components g_{ij} (resp. $g_{\alpha\beta}$) given by (1.18a) (resp. (1.24)).

Proposition 1.8. *The structural and transversal Vrănceanu covariant derivatives of g_{ij} and $g_{\alpha\beta}$ are given by*

$$\begin{gathered}\text{(a) } g_{ij\|k} = 0, \quad \text{(b) } g_{ij|\gamma} = \frac{\delta g_{ij}}{\delta x^\gamma} - g_{hj}\frac{\partial A^h_\gamma}{\partial x^i} - g_{ih}\frac{\partial A^h_\gamma}{\partial x^j}, \\ \text{(c) } g^{ij}{}_{\|k} = 0,\end{gathered} \tag{1.41}$$

and

$$\text{(a) } g_{\alpha\beta\|k} = \frac{\partial g_{\alpha\beta}}{\partial x^k}, \quad \text{(b) } g_{\alpha\beta|\gamma} = 0, \quad \text{(c) } g^{\alpha\beta}{}_{|\gamma} = 0, \tag{1.42}$$

respectively.

Proof. We replace $\{QX, QY, QZ\}$ in (1.13a) by $\left\{\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right\}$ and by using (1.18a), (1.22a) and (1.35) for g_{ij}, we obtain (1.41a). In a similar way (1.42b) follows from (1.13b). Next, we apply (1.34) for g_{ij} and by using (1.25b) we infer (1.41b). Also, from (1.36), (1.42a) follows. Finally, (1.41c) and (1.42c) are consequences of (1.41a) and (1.42b) respectively. ■

We note that (1.41b) is more complicated than all the other covariant derivatives. For this reason we present an equivalent formula to (1.41b). First we define the local functions

$$\Gamma_{i\alpha j} = \frac{1}{2}\left(\frac{\partial g_{i\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\alpha}\right), \tag{1.43}$$

where g_{ij} and $g_{i\alpha}$ are given by (1.18). Then we state the following.

Proposition 1.9. *The transversal Vrănceanu covariant derivative of* g_{ij} *is given by*

$$g_{ij|\gamma} = 2\left(C_i{}^k{}_j\ g_{k\alpha} - \Gamma_{i\gamma j}\right). \tag{1.44}$$

Proof. Take the partial derivatives of $g_{ih}A^h_\gamma = g_{i\gamma}$ with respect to x^j and obtain

$$g_{ih}\ \frac{\partial A^h_\gamma}{\partial x^j} = \frac{\partial g_{i\gamma}}{\partial x^j} - A^h_\gamma\ \frac{\partial g_{ih}}{\partial x^j}\,. \tag{1.45}$$

Then by using (1.21), (1.45) and (1.43) in (1.41b) we deduce that

$$\begin{aligned} g_{ij|\gamma} &= \frac{\partial g_{ij}}{\partial x^\gamma} - A^h_\gamma\ \frac{\partial g_{ij}}{\partial x^h} + A^h_\gamma\ \frac{\partial g_{jh}}{\partial x^i} - \frac{\partial g_{j\gamma}}{\partial x^i} + A^h_\gamma\ \frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{i\gamma}}{\partial x^j} \\ &= A^h_\gamma\left(\frac{\partial g_{jh}}{\partial x^i} + \frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^h}\right) - 2\Gamma_{i\gamma j}. \end{aligned} \tag{1.46}$$

Thus (1.44) follows from (1.46) by using (1.20) and (1.25a). ∎

Finally, from Section 2.4 we derive the Ricci and Bianchi identities for the Vrănceanu connection. First, we use (1.27) and (1.33) in (2.4.6)–(2.4.8) and (2.4.14)–(2.4.16), and deduce that the **structural** and **transversal Ricci identities** for ∇^* are given by:

$$U^i{}_{|\alpha|\beta} - U^i{}_{|\beta|\alpha} = U^j R_j{}^i{}_{\alpha\beta} - U^i{}_{\|k} T^*{}_\alpha{}^k{}_\beta, \tag{1.47}$$

$$U^i{}_{|\alpha\|j} - U^i{}_{\|j|\alpha} = U^h R_h{}^i{}_{\alpha j}, \tag{1.48}$$

$$U^i{}_{\|j\|k} - U^i{}_{\|k\|j} = U^h R_h{}^i{}_{jk}, \tag{1.49}$$

and

$$Z^\alpha{}_{|\beta|\gamma} - Z^\alpha{}_{|\gamma|\beta} = Z^\varepsilon R'_\varepsilon{}^\alpha{}_{\beta\gamma} - \frac{\partial Z^\alpha}{\partial x^k}\ T^*{}_\beta{}^k{}_\gamma, \tag{1.50}$$

$$Z^\alpha{}_{|\beta\|j} - Z^\alpha{}_{\|j|\beta} = Z^\varepsilon R'_\varepsilon{}^\alpha{}_{\beta j}, \tag{1.51}$$

$$Z^\alpha{}_{\|j\|k} - Z^\alpha{}_{\|k\|j} = 0, \tag{1.52}$$

respectively.

Next, by using (1.27) and (1.26a) in (2.4.22), (2.4.24), (2.4.26), (2.4.28) and (2.4.29)–(2.4.32), we obtain the following **structural Bianchi identities** for the Vrănceanu connection:

$$\sum_{(\alpha,\beta,\gamma)} \left\{ T^{*}{}_{\alpha}{}^{i}{}_{\beta|\gamma} \right\} = 0, \tag{1.53}$$

$$T^{*}{}_{\alpha}{}^{i}{}_{\beta\|k} = R_{k}{}^{i}{}_{\alpha\beta}, \tag{1.54}$$

$$R_{j}{}^{i}{}_{\alpha k} = R_{k}{}^{i}{}_{\alpha j}, \tag{1.55}$$

$$\sum_{(i,j,k)} \left\{ R_{i}{}^{h}{}_{jk} \right\} = 0, \tag{1.56}$$

$$\sum_{(\alpha,\beta,\gamma)} \left\{ R_{i}{}^{h}{}_{\alpha\beta|\gamma} + R_{i}{}^{h}{}_{\alpha j}\ T^{*}{}_{\beta}{}^{j}{}_{\gamma} \right\} = 0, \tag{1.57}$$

$$R_{i}{}^{h}{}_{\beta\gamma\|j} + R_{i}{}^{h}{}_{\gamma j|\beta} - R_{i}{}^{h}{}_{\beta j|\gamma} + R_{i}{}^{h}{}_{jk}\ T^{*}{}_{\beta}{}^{k}{}_{\gamma} = 0, \tag{1.58}$$

$$R_{i}{}^{h}{}_{jk|\gamma} + R_{i}{}^{h}{}_{\gamma j\|k} - R_{i}{}^{h}{}_{\gamma k\|j} = 0, \tag{1.59}$$

$$\sum_{(j,k,h)} \left\{ R_{i}{}^{r}{}_{jk\|h} \right\} = 0. \tag{1.60}$$

Similarly, by using (1.27), (1.26a) and (1.33) in (2.4.23), (2.4.25), (2.4.27) and (2.4.33)–(2.4.36) we deduce the following **transversal Bianchi identities** for the Vrănceanu connection:

$$\sum_{(\alpha,\beta,\gamma)} \{ R'{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma} \} = 0, \tag{1.61}$$

$$R'{}_{\alpha}{}^{\gamma}{}_{\beta k} = R'{}_{\beta}{}^{\gamma}{}_{\alpha k}, \tag{1.62}$$

$$\sum_{(\alpha,\beta,\gamma)} \left\{ R'{}_{\mu}{}^{\varepsilon}{}_{\alpha\beta|\gamma} + R'{}_{\mu}{}^{\varepsilon}{}_{\gamma i}\ T^{*}{}_{\alpha}{}^{i}{}_{\beta} \right\} = 0, \tag{1.63}$$

$$R'{}_{\mu}{}^{\nu}{}_{\beta\gamma\|j} + R'{}_{\mu}{}^{\nu}{}_{\gamma j|\beta} - R'{}_{\mu}{}^{\nu}{}_{\beta j|\gamma} = 0, \tag{1.64}$$

$$R'{}_{\mu}{}^{\nu}{}_{\gamma j\|k} = R'{}_{\mu}{}^{\nu}{}_{\gamma k\|j}. \tag{1.65}$$

Because of (1.26a) and (1.33) the identities (2.4.27) and (2.4.36) become trivial for the Vrănceanu connection. All these Ricci and Bianchi identities have been obtained by the authors in Bejancu–Farran [BF03a].

Remark 1.3. The above Bianchi identities shed more light on the curvature tensor field of the Vrănceanu connection. For example, the identity (1.54) gives an elegant formula for $R_{k}{}^{i}{}_{\alpha\beta}$ (compare with (1.28)). Also, from (1.55) and (1.62) we deduce that $R_{j}{}^{i}{}_{\alpha k}$ and $R'{}_{\alpha}{}^{\gamma}{}_{\beta k}$ are symmetric adapted tensor fields with respect to indices (jk) and $(\alpha\beta)$ respectively. ■

3.2 The Schouten–Van Kampen Connection on a Foliated Semi–Riemannian Manifold

Let $(M, g, \mathcal{F})$ be an $(n+p)$–dimensional foliated semi–Riemannian manifold with structural and transversal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ of rank n and p respectively. In this section we develop a study that is inspired by the theory of non–degenerate submanifolds of semi–Riemannian manifolds (cf. O'Neill [O83], p. 97), and obtain some induced geometrical objects on both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$. In particular, the pair of induced connections $(\nabla, \nabla^{\perp})$ determines the Schouten–Van Kampen connection induced by the Levi–Civita connection on (M, g) (see Section1.5).

Let $\widetilde{\nabla}$ be the Levi–Civita connection on (M, g). Then according to the theory we developed in Section 1.5 (see (1.5.17)–(1.5.20)) we have

$$\widetilde{\nabla}_X QY = \nabla_X QY + h(X, QY), \tag{2.1}$$

and

$$\widetilde{\nabla}_X Q'Y = h'(X, QY) + \nabla^{\perp}_X Q'Y, \tag{2.2}$$

where we set:

$$\text{(a) } \nabla_X QY = Q\widetilde{\nabla}_X QY, \quad \text{(b) } \nabla^{\perp}_X Q'Y = Q'\widetilde{\nabla}_X Q'Y, \tag{2.3}$$

and

$$\text{(a) } h(X, QY) = Q'\widetilde{\nabla}_X QY, \quad \text{(b) } h'(X, Q'Y) = Q\widetilde{\nabla}_X Q'Y, \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$. Here, ∇ and $\nabla^{\perp}$ are the induced connections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. Also, we call $h : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}^{\perp})$ given by

$$h(QX, QY) = Q'\widetilde{\nabla}_{QX} QY, \quad \forall\, X, Y \in \Gamma(TM), \tag{2.5}$$

the **second fundamental form** of the foliation $\mathcal{F}$. Clearly, at any point $x \in M$, h coincides with the second fundamental form of the leaf of $\mathcal{F}$ passing through x. Similarly, we call $h' : \Gamma(\mathcal{D}^{\perp}) \times \Gamma(\mathcal{D}^{\perp}) \longrightarrow \Gamma(\mathcal{D})$ defined by

$$h'(Q'X, Q'Y) = Q\widetilde{\nabla}_{Q'X} Q'Y, \quad \forall\, X, Y \in \Gamma(TM), \tag{2.6}$$

the **second fundamental form** of the transversal distribution $\mathcal{D}^{\perp}$.

Lemma 2.1. *The induced geometric objects $\nabla, \nabla^{\perp}$, h and h' satisfy the following equalities:*

$$\begin{aligned}
&\text{(a) } \nabla_{QX}QY - \nabla_{QY}QX - [QX, QY] = 0,\\
&\text{(b) } h(QX, QY) = h(QY, QX),\\
&\text{(c) } \nabla_{Q'X}QY - D_{Q'X}QY = h'(QY, Q'X),\\
&\text{(d) } \nabla^{\perp}_{QY}Q'X - D^{\perp}_{QY}Q'X = h(Q'X, QY),\\
&\text{(e) } h'(Q'X, Q'Y) - h'(Q'Y, Q'X) = Q[Q'X, Q'Y],\\
&\text{(f) } \nabla^{\perp}_{Q'X}Q'Y - \nabla^{\perp}_{Q'Y}Q'X = Q'[Q'X, Q'Y],\\
&\text{(g) } \nabla_{QX}QY = D_{QX}QY,\\
&\text{(h) } \nabla^{\perp}_{Q'X}Q'Y = D^{\perp}_{Q'X}Q'Y,\\
&\text{(k) } (\nabla_X g)(QY, QZ) = X(g(QY, QZ)) - g(\nabla_X QY, QZ)\\
&\qquad\qquad -g(QY, \nabla_X QZ) = 0,\\
&(\ell)\ (\nabla^{\perp}_X g)(Q'Y, Q'Z) = X(g(Q'Y, Q'Z)) - g(\nabla^{\perp}_X Q'Y, Q'Z)\\
&\qquad\qquad -g(Q'Y, \nabla^{\perp}_X Q'Z) = 0,
\end{aligned} \tag{2.7}$$

for any $X, Y, Z \in \Gamma(TM)$, where D and $D^{\perp}$ are the intrinsic connections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

Proof. By direct calculations using (2.3), (2.4), (1.7), (1.9), (1.10) and taking into account that $\mathcal{D}$ is integrable and $\widetilde{\nabla}$ satisfies (1.5.8) and (1.5.9). ■

Corollary 2.2. *Let $(M, g, \mathcal{F})$ be a foliated semi–Riemannian manifold. Then we have the assertions:*

(i) *The second fundamental form of the foliation is symmetric.*
(ii) *The second fundamental form of the transversal distribution is symmetric if and only if $\mathcal{D}^{\perp}$ is integrable.*

Next, from (2.1) and (2.2) we obtain

$$\begin{aligned}
&\text{(a) } \widetilde{\nabla}_{QX}QY = \nabla_{QX}QY + h(QX, QY),\\
&\text{(b) } \widetilde{\nabla}_{QX}Q'Y = -A_{Q'Y}QX + \nabla^{\perp}_{QX}Q'Y,
\end{aligned} \tag{2.8}$$

where $A_{Q'Y} : \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ is an $F(M)$–linear operator given by

$$A_{Q'Y}QX = -h'(QX, Q'Y) = -Q\widetilde{\nabla}_{QX}Q'Y. \tag{2.9}$$

According to the terminology from the theory of submanifolds we call $A_{Q'Y}$ the **shape operator** of the foliation $\mathcal{F}$ with respect to the normal section $Q'Y$. Similarly, we write:

$$\begin{aligned}
&\text{(a) } \widetilde{\nabla}_{Q'X}Q'Y = h'(Q'X, Q'Y) + \nabla^{\perp}_{Q'X}Q'Y,\\
&\text{(b) } \widetilde{\nabla}_{Q'X}QY = \nabla_{Q'X}QY - A'_{QY}Q'X,
\end{aligned} \tag{2.10}$$

where $A'_{QY} : \Gamma(\mathcal{D}^\perp) \to \Gamma(\mathcal{D}^\perp)$ is an $F(M)$–linear operator given by

$$A'_{QY}Q'X = -h(Q'X, QY) = -Q'\widetilde{\nabla}_{Q'X}QY. \tag{2.11}$$

Then we call A'_{QY} the **shape operator** of the transversal distribution with respect to $QY \in \Gamma(\mathcal{D})$.

Taking into account that h is symmetric (see (2.7b)) and by using (1.5.23)–(1.5.25) and the assertion (iii) of Lemma 1.5.5 we state the following:

Lemma 2.3. *The second fundamental forms and the shape operators of $\mathcal{F}$ and $\mathcal{D}^\perp$ satisfy:*

$$\begin{aligned} &(a)\ g(h(QX,QY),Q'Z) + g(h'(QX,Q'Z),QY) = 0,\\ &(b)\ g(h'(Q'X,Q'Y),QZ) + g(h(Q'X,QZ),Q'Y) = 0,\\ &(c)\ g(A_{Q'Z}QX,QY) = g(QX,A_{Q'Z}QY) = g(h(QX,QY),Q'Z),\\ &(d)\ g(A'_{QZ}Q'X,Q'Y) = g(h'(Q'X,Q'Y),QZ), \end{aligned} \tag{2.12}$$

for any $X, Y, Z \in \Gamma(TM)$.

Corollary 2.4.
(i) *The shape operator of the foliation $\mathcal{F}$ is self–adjoint.*
(ii) *The shape operator of the transversal distribution is self–adjoint if and only if $\mathcal{D}^\perp$ is integrable.*

The basic properties of foliations with special second fundamental forms are presented in Section 3.4.

Next, we denote by ∇° the Schouten–Van Kampen connection determined by the Levi–Civita connection $\widetilde{\nabla}$ on (M, g), that is, we have (cf. (1.3.15))

$$\nabla^\circ_X Y = Q\widetilde{\nabla}_X QY + Q'\widetilde{\nabla}_X Q'Y, \quad \forall\, X, Y \in \Gamma(TM). \tag{2.13}$$

Remark 2.1. From (2.13) we can see that the **almost product connection** defined by Reinhart [Rei83], p. 147 is just the Schouten–Van Kampen connection. ■

Next, from Theorem 1.5.7 it follows that ∇° is an adapted linear connection on $(M, \mathcal{D}, \mathcal{D}^\perp)$ determined by the pair of induced connections $(\nabla, \nabla^\perp)$. Hence we have:

$$\nabla^\circ_X Y = \nabla_X QY + \nabla^\perp_X Q'Y, \quad \forall\, X, Y \in \Gamma(TM). \tag{2.14}$$

Taking into account that on $(M, g, \mathcal{F})$ we also constructed the Vrănceanu connection ∇^*, we should investigate the case $\nabla^\circ = \nabla^*$. This was done in a more general setting in Section 1.5 for two complementary orthogonal semi–Riemannian distributions. Thus we only recall here the following important result (see Theorem 1.5.8).

Theorem 2.5. *Let $(M, g, \mathcal{F})$ be a foliated semi–Riemannian manifold. Then the Schouten–Van Kampen and Vrănceanu connections coincide if and only if $\mathcal{D}^\perp$ is integrable and M is a locally semi–Riemannian product of local leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$.*

Now, we find the local coefficients for the Schouten–Van Kampen connection. First, we put:

$$\begin{aligned} &\text{(a) } \nabla^\circ_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = C^\circ{}_i{}^k{}_j \, \frac{\partial}{\partial x^k} \,, \\ &\text{(b) } \nabla^\circ_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = \nabla_{\frac{\delta}{\delta x^\alpha}} \frac{\partial}{\partial x^i} = D^\circ{}_i{}^k{}_\alpha \, \frac{\partial}{\partial x^k} \,, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\text{(a) } \nabla^\circ_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = \nabla^\perp_{\frac{\partial}{\partial x^i}} \frac{\delta}{\delta x^\alpha} = L^\circ{}_\alpha{}^\beta{}_i \, \frac{\delta}{\delta x^\beta} \,, \\ &\text{(b) } \nabla^\circ_{\frac{\delta}{\delta x^\gamma}} \frac{\delta}{\delta x^\alpha} = \nabla^\perp_{\frac{\delta}{\delta x^\gamma}} \frac{\delta}{\delta x^\alpha} = F^\circ{}_\alpha{}^\beta{}_\gamma \, \frac{\delta}{\delta x^\beta} \,. \end{aligned} \tag{2.16}$$

Also we need some local components for the bilinear mappings h and h':

$$\begin{aligned} &\text{(a) } h\left(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^i}\right) = h_\alpha{}^\beta{}_i \, \frac{\delta}{\delta x^\beta}, \\ &\text{(b) } h'\left(\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right) = h'{}_i{}^k{}_\alpha \, \frac{\partial}{\partial x^k} \,. \end{aligned} \tag{2.17}$$

Proposition 2.6. *The local coefficients of the induced connections ∇ and $\nabla^\perp$ with respect to the semi–holonomic frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ are given by*

$$\begin{aligned} &\text{(a) } C^\circ{}_i{}^k{}_j = C_i{}^k{}_j = \frac{1}{2}\, g^{kh}\left(\frac{\partial g_{hi}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right), \\ &\text{(b) } D^\circ{}_i{}^k{}_\alpha = \frac{1}{2}\, g^{kj}\left(\frac{\delta g_{ij}}{\delta x^\alpha} + D_i{}^k{}_\alpha \, g_{kj} - D_j{}^k{}_\alpha \, g_{ki}\right) \\ &\qquad\qquad = D_i{}^k{}_\alpha + h'{}_i{}^k{}_\alpha, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &\text{(a) } L^\circ{}_\alpha{}^\beta{}_i = \frac{1}{2}\, g^{\beta\gamma}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^i} - T^*{}_\alpha{}^k{}_\gamma \, g_{ki}\right) = h_\alpha{}^\beta{}_i, \\ &\text{(b) } F^\circ{}_\alpha{}^\beta{}_\gamma = F_\alpha{}^\beta{}_\gamma = \frac{1}{2}\, g^{\beta\mu}\left(\frac{\delta g_{\mu\alpha}}{\delta x^\gamma} + \frac{\delta g_{\mu\gamma}}{\delta x^\alpha} - \frac{\delta g_{\alpha\gamma}}{\delta x^\mu}\right), \end{aligned} \tag{2.19}$$

respectively, where $(C_i{}^k{}_j, D_i{}^k{}_\alpha, F_\alpha{}^\beta{}_\gamma)$ are the local coefficients of ∇^* and $T^*{}_\alpha{}^k{}_\beta$ is the integrability tensor of $\mathcal{D}^\perp$.

Proof. By using (2.7g), (2.7h), (2.15a) and (2.16b) we obtain (2.18a) and (2.19b). Next, we use (2.3), (1.5.10), (2.15b), (2.16a), (2.2.18), (2.3.21), and we deduce the first equalities in (2.18b) and (2.19a). Finally, the second equalities in (2.18b) and (2.19a) follow by using (2.7c), (2.7d), (2.15b), (2.16a), (1.22b), (1.23a) and (2.17). ∎

Corollary 2.7. *The local coefficients of the Schouten–Van Kampen connection with respect to the semi–holonomic frame field* $\left\{ \frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha} \right\}$ *are given by* (2.18) *and* (2.19).

As a consequence of (2.7ℓ) and (2.7k) we state the following.

Proposition 2.8. *The Schouten–Van Kampen connection* ∇° *is a metric adapted linear connection on* $(M, g, \mathcal{F})$*, that is, we have*

$$\begin{aligned} &(a)\ g_{ij\|^\circ k} = 0, \\ &(b)\ g_{ij|^\circ \gamma} = 0, \\ &(c)\ g_{\alpha\beta\|^\circ k} = 0, \\ &(d)\ g_{\alpha\beta|^\circ \gamma} = 0. \end{aligned} \tag{2.20}$$

where we denoted by $|^\circ$ *and* $\|^\circ$ *the transversal and structural covariant derivatives with respect to Schouten–Van Kampen connection.*

Also, by using (2.3.40), (2.18), (2.19) and (1.25b) we obtain the following.

Proposition 2.9. *The local components of the torsion tensor field* T° *of the Schouten–Van Kampen connection are given by*

$$\begin{aligned} &(a)\ T^{\circ}{}_{i}{}^{k}{}_{j} = 0, \\ &(b)\ T^{\circ}{}_{i}{}^{k}{}_{\alpha} = -T^{\circ}{}_{\alpha}{}^{k}{}_{i} = h'{}_{i}{}^{k}{}_{\alpha}, \\ &(c)\ T^{\circ}{}_{\alpha}{}^{\gamma}{}_{\beta} = 0, \\ &(d)\ T^{\circ}{}_{\alpha}{}^{\beta}{}_{i} = L^{\circ}{}_{\alpha}{}^{\beta}{}_{i} = h_{\alpha}{}^{\beta}{}_{i}, \\ &(e)\ T^{\circ}{}_{\alpha}{}^{k}{}_{\beta} = T^{*}{}_{\alpha}{}^{k}{}_{\beta} = \frac{\delta A^k_\alpha}{\delta x^\beta} - \frac{\delta A^k_\beta}{\delta x^\alpha}. \end{aligned} \tag{2.21}$$

Finally, by using (2.3.32)–(2.3.37), (2.18), (2.19), (1.28)–(1.33), (2.3.17) and (2.3.18) we deduce all the local components of R° as they are stated in the next proposition.

Proposition 2.10. *The local components of the curvature tensor field* R° *of the Schouten–Van Kampen connection are given by*

$$
\begin{aligned}
&\text{(a) } R^{\circ}{}_{i}{}^{t}{}_{\alpha\beta}=R_{i}{}^{t}{}_{\alpha\beta}+h'{}_{i}{}^{t}{}_{\alpha|\beta}-h'{}_{i}{}^{t}{}_{\beta|\alpha}+h'{}_{i}{}^{j}{}_{\alpha}\ h'{}_{j}{}^{t}{}_{\beta}-h'{}_{i}{}^{j}{}_{\beta}\ h'{}_{j}{}^{t}{}_{\alpha},\\
&\text{(b) } R^{\circ}{}_{i}{}^{t}{}_{\alpha k}=R_{i}{}^{t}{}_{\alpha k}+h'{}_{i}{}^{t}{}_{\alpha\|k},\\
&\text{(c) } R^{\circ}{}_{i}{}^{t}{}_{jk}=R_{i}{}^{t}{}_{jk},\\
&\text{(d) } R^{\circ}{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}=R'{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}+h_{\alpha}{}^{\varepsilon}{}_{j}\ T^{*}{}_{\beta}{}^{j}{}_{\gamma},\\
&\text{(e) } R^{\circ}{}_{\alpha}{}^{\varepsilon}{}_{\beta i}=R'{}_{\alpha}{}^{\varepsilon}{}_{\beta i}-h_{\alpha}{}^{\varepsilon}{}_{i|\beta},\\
&\text{(f) } R^{\circ}{}_{\alpha}{}^{\varepsilon}{}_{ij}=\frac{\partial h_{\alpha}{}^{\varepsilon}{}_{i}}{\partial x^{j}}-\frac{\partial h_{\alpha}{}^{\varepsilon}{}_{j}}{\partial x^{i}}+h_{\alpha}{}^{\beta}{}_{i}\ h_{\beta}{}^{\varepsilon}{}_{j}-h_{\alpha}{}^{\beta}{}_{j}\ h_{\beta}{}^{\varepsilon}{}_{i},
\end{aligned}
\tag{2.22}
$$

where the terms appearing on the right hand side of these equations are the local components of the torsion and curvature tensor fields of the Vrănceanu connection ∇^*, and all covariant derivatives are considered with respect to ∇^*.

3.3 Foliated Semi–Riemannian Manifolds with Bundle–Like Metrics

The purpose of this section is to study the geometry of foliations with bundle–like metrics on semi–Riemannian manifolds. This important class of foliations was introduced by Reinhart [Rei59a] in the Riemannian case. First we introduce those foliations and then we find several of their geometric characterizations. This is followed by determining explicit expressions for the local components of the curvature tensor of the intrinsic connection $D^{\perp}$ on $\mathcal{D}^{\perp}$, and for the transversal Bianchi identities with respect to the Vrănceanu connection. It is noteworthy that the curvature tensor field of $D^{\perp}$ satisfies the same identities as the curvature tensor field of the Levi–Civita connection. This enables us to define and study foliated semi–Riemannian manifolds of constant transversal Vrănceanu curvature and transversal Einstein foliated semi–Riemannian manifolds.

Let $(M, g, \mathcal{F})$ be an $(n+p)$–dimensional foliated semi–Riemannian manifold, where $\mathcal{F}$ is a non–degenerate n–foliation. Consider the intrinsic connection $D^{\perp}$ on the transversal distribution $\mathcal{D}^{\perp}$ (see (1.8) and (1.9)), and give the following definition. We say that the semi–Riemannian metric g on M is **bundle–like** for the non–degenerate foliation $\mathcal{F}$ if the induced semi–Riemannian metric on $\mathcal{D}^{\perp}$ by g (denoted by the same symbol g) is parallel with respect to the intrinsic connection $D^{\perp}$, that is, we have (see (1.5.28))

$$
\begin{aligned}
(D^{\perp}_{X}g)(Q'Y, Q'Z) = {} & X(g(Q'Y, Q'Z)) - g(D^{\perp}_{X}Q'Y, Q'Z)\\
& - g(Q'Y, D^{\perp}_{X}Q'Z) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM).
\end{aligned}
\tag{3.1}
$$

When for a given foliation $\mathcal{F}$ there exists a semi–Riemannian (Riemannian) metric g on M which is bundle–like for $\mathcal{F}$, we say that $\mathcal{F}$ is a **semi–Riemannian** (**Riemannian**) **foliation** on (M, g), and g is bundle–like for $\mathcal{F}$. An explanation of the above name for $\mathcal{F}$ is given later on in this section.

Comparing with the terminology we introduced in Section 1.7 on non–holonomic manifolds, we see that g is bundle–like if and only if g is Vrănceanu–parallel on $\mathcal{D}^{\perp}$.

Moreover, taking into account (1.5), (1.17) and (3.1) we state the following.

Theorem 3.1. *The semi–Riemannian metric g on M is bundle–like for $\mathcal{F}$ if and only if we have*

$$QX(g(Q'Y, Q'Z)) - g([QX, Q'Y], Q'Z) - g([QX, Q'Z], Q'Y) = 0, \tag{3.2}$$

for any $X, Y, Z \in \Gamma(TM)$.

Remark 3.1. By using the metric isomorphism $\mathcal{D}^{\perp} \approx TM/\mathcal{D}$ it is easy to see that the characterization of a bundle–like metric stated in Theorem 3.1 coincides with the one presented in Tondeur [Ton97], p. 43, for a Riemannian metric. Also, in the above reference, the foliation $\mathcal{F}$ is called a Riemannian foliation or a foliation with holonomy invariant transversal bundle. ■

Since it was introduced by Reinhart, the class of foliations with bundle–like metrics on Riemannian manifolds was the focus of investigation and attention of many geometers. Several interesting results appeared. We will not present all those results here, but we refer the reader to Tondeur for references to the original papers.

Our definition of foliations with bundle–like metric is not the definition given originally by Reinhart. However, those two definitions are equivalent as we see below.

To reach the original definition given by Reinhart, we consider locally a semi–holonomic frame field $\left\{ \frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha} \right\}$ on $(M, g, \mathcal{F})$, where $\frac{\delta}{\delta x^\alpha}$ are given by (1.21). Then, by using the dual semi–holonomic frame field $\{\delta x^i, dx^\alpha\}$, where we put

$$\delta x^i = dx^i + A^i_\alpha \, dx^\alpha, \tag{3.3}$$

we obtain the following local expression for the semi–Riemannian metric g:

$$g_{\text{local}} = g_{ij}(x^k, x^\gamma)\delta x^i \, \delta x^j + g_{\alpha\beta}(x^k, x^\gamma) dx^\alpha \, dx^\beta, \tag{3.4}$$

where g_{ij} and $g_{\alpha\beta}$ are defined by (1.18a) and (1.24) respectively.

Theorem 3.2. *The semi–Riemannian metric g on M is bundle–like if and only if the transversal local components $g_{\alpha\beta}$ of g define a basic transversal tensor field, that is, we have*

$$\frac{\partial g_{\alpha\beta}}{\partial x^i} = 0, \quad \forall\, i \in \{1, ..., n\}, \;\; \alpha, \beta \in \{n+1, ..., n+p\}. \tag{3.5}$$

Proof. Replace $\{QX, Q'Y, Q'Z\}$ from (3.2) by $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right\}$ and by using (1.24) we obtain

$$\frac{\partial g_{\alpha\beta}}{\partial x^i} - g\left(\left[\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right], \frac{\delta}{\delta x^\beta}\right) - g\left(\left[\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\beta}\right], \frac{\delta}{\delta x^\alpha}\right) = 0.$$

Taking into account that the above Lie brackets do not have transversal components (see (2.3.21)), we deduce that (3.2) and (3.5) are equivalent. ■

Remark 3.2. The condition (3.5) for the semi–Riemannian metric g represents the definition given by Reinhart [Rei59a], p.122, for Riemannian bundle–like metrics. ■

Remark 3.3. By using (3.5) we also see that g is bundle–like for $\mathcal{F}$ if and only if the transversal tensor field $g_{\alpha\beta}$ is structural Vrănceanu parallel. ■

Due to (3.4) and Theorem 3.2 we deduce that the local expression for the bundle–like semi–Riemannian metric g is the following

$$g_{\text{local}} = g_{ij}(x^k, x^\gamma)\delta x^i\, \delta x^j + g_{\alpha\beta}(x^\gamma) dx^\alpha\, dx^\beta. \tag{3.6}$$

Remark 3.4. An intuitive geometrical meaning of a bundle–like Riemannian metric g was given by Reinhart [Rei59a], p. 123. Namely, he proved that g is bundle–like if and only if each geodesic in (M, g) which is tangent to $\mathcal{D}^\perp$ at one point remains tangent for its entire length. This characterization of a bundle–like Riemannian metric gives a reason for the name **totally geodesic distribution** for $\mathcal{D}^\perp$ (cf. Reinhart [Rei83], p. 150). When the leaves of the foliation are totally geodesic immersed in (M, g), Yorozu [Y83] proved that g is bundle–like if and only if all geodesics in M make a constant angle with leaves. ■

Several characterizations of bundle–like metrics are presented in the next theorem.

Theorem 3.3. *Let $(M, g, \mathcal{F})$ be a foliated semi–Riemannian manifold, where $\mathcal{F}$ is a non–degenerate foliation. Then the following assertions are equivalent:*

(i) *g is a bundle–like metric for $\mathcal{F}$.*

(ii) *The induced metric g on $\mathcal{D}^\perp$ is parallel with respect to Vrănceanu connection ∇^*.*

(iii) *The Levi–Civita connection $\widetilde{\nabla}$ on (M, g) satisfies any one (and hence all) of the following equalities:*

$$g(\widetilde{\nabla}_{Q'Y} QX, Q'Z) + g(\widetilde{\nabla}_{Q'Z} QX, Q'Y) = 0, \tag{3.7}$$

$$g(QX, \widetilde{\nabla}_{Q'Y} Q'Z + \widetilde{\nabla}_{Q'Z} Q'Y) = 0, \tag{3.8}$$

$$2g(\widetilde{\nabla}_{Q'Y} Q'Z, QX) = g([Q'Y, Q'Z], QX), \tag{3.9}$$

(iv) *QX is a $\mathcal{D}^{\perp}$–Killing vector field for any $X \in \Gamma(TM)$.*
(v) *The second fundamental form h' of $\mathcal{D}^{\perp}$ is given by*

$$h'(Q'Y, Q'Z) = \frac{1}{2} Q[Q'Y, Q'Z], \quad \forall Y, Z \in \Gamma(TM). \tag{3.10}$$

(vi) *The symmetric second fundamental form h'^{s} of $\mathcal{D}^{\perp}$ vanishes identically on M.*
(vii) *For any $X \in \Gamma(TM)$ the shape operator A'_{QX} of $\mathcal{D}^{\perp}$ is skew–symmetric with respect to g, that is, we have*

$$g(A'_{QX}Q'Y, Q'Z) + g(Q'Y, A'_{QX}Q'Z) = 0, \quad \forall Y, Z \in \Gamma(TM). \tag{3.11}$$

(viii) *The torsion tensor field of Vrănceanu connection ∇^* is given by*

$$T^*(X, Y) = -2h'(Q'X, Q'Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.12}$$

Proof. The equivalence of (i) and (ii) follows by using (3.1) and (1.11). By using (1.5.8) and (1.5.9) for $\widetilde{\nabla}$ in (3.2) we deduce that (3.2) and (3.7) are equivalent. The same conditions (1.5.8) and (1.5.9) for $\widetilde{\nabla}$ imply the equivalence of (3.7), (3.8) and (3.9). Thus (i) and (iii) are equivalent. By using (1.5.35) for $\mathcal{D}^{\perp}$ we infer the equivalence of (3.7) and (iv) and therefore of (iii) and (iv). Next, by using (2.6) and (3.9) we obtain

$$g(h'(Q'Y, Q'Z), QX) = g(Q\widetilde{\nabla}_{Q'Y}Q'Z, QX) = \frac{1}{2} g(Q[Q'Y, Q'Z], QX),$$

which proves the equivalence of (3.9) and (3.10). Taking into account (1.5.34) we deduce that the symmetric second fundamental form h'^{s} of $\mathcal{D}^{\perp}$ is given by

$$h'^{s}(Q'Y, Q'Z) = \frac{1}{2}\left(h'(Q'Y, Q'Z) + h'(Q'Z, Q'Y)\right), \quad \forall Y, Z \in \Gamma(TM). \tag{3.13}$$

Then by using (2.6), (3.8) and (3.13) we deduce the equivalence of (vi) and (iii). Clearly, (vi) and (vii) are equivalent via (2.12d). Finally, since h is symmetric, by using (1.6.14) we obtain the equivalence of (vi) and (viii). ∎

Theorem 3.4. *Let $(M, g, \mathcal{F})$ be a foliated semi–Riemannian manifold, where $\mathcal{F}$ is a non–degenerate foliation and g is bundle–like for $\mathcal{F}$. Then the following assertions are equivalent:*

(i) *$\mathcal{D}^{\perp}$ is an integrable distribution.*
(ii) *The second fundamental form h' of $\mathcal{D}^{\perp}$ vanishes identically on M, that is, we have*

$$h'(Q'X, Q'Y) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{3.14}$$

(iii) *The* $F(M)$*-bilinear mapping* h *given by* (2.4a) *satisfies*

$$h(Q'X, QY) = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{3.15}$$

Proof. The equivalence of (i) and (ii) follows from (3.10). Next, by using (2.12b) we obtain

$$g(h(Q'X, QY), Q'Z) + g(h'(Q'X, Q'Z), QY) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM),$$

which proves that (ii) and (iii) are equivalent. ∎

Examples of foliations with bundle–like metric on Riemannian (semi–Riemannian) manifolds are abundent. Here we present some of them.

Example 3.5. Let X be a non–zero Killing vector field on a semi–Riemannian manifold (M, g), that is $\mathcal{L}_X\, g = 0$ where $\mathcal{L}$ is the Lie derivative. This is equivalent to saying that X and g satisfy (cf. O'Neill [O83], p. 251)

$$g(\widetilde{\nabla}_Y X, Z) + g(\widetilde{\nabla}_Z X, Y) = 0, \quad \forall\, Y, Z \in \Gamma(TM),$$

where $\widetilde{\nabla}$ is the Levi–Civita connection on (M, g). Then the flow of X defines a bundle–like foliation on (M, g). This follows from the assertion (iv) of Theorem 3.3, taking into account that any Killing vector field is $\mathcal{D}^{\perp}$–Killing. ∎

Example 3.6. Let M be the total space of a fiber bundle over a semi–Riemannian manifold (N, h). Denote by $\mathcal{F}$ the foliation by components of fibers of M (see Example 2.1.4), and by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$. Let $\mathcal{D}'$ be a transversal distribution to $\mathcal{D}$ and k be a semi–Riemannian metric on $\mathcal{D}$. The paracompactness of M and N guarantees the existence of $\mathcal{D}'$ and of a Riemannian metric k on $\mathcal{D}$. Let $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$ be a foliated chart on M which induces the semi–holonomic frame field $\left\{ \frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha} \right\}$, $i \in \{1, ..., n\}$, $\alpha \in \{n+1, ..., n+p\}$. As in this case (x^α) are the local coordinates on N, the local components of h are given by

$$h_{\alpha\beta}(x^{n+1}, ..., x^{n+p}) = h\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right).$$

Also, since $\frac{\partial}{\partial x^i} \in \Gamma(\mathcal{D})$ we put

$$k_{ij}(x^i, x^\alpha) = k\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right).$$

Then we define a semi–Riemannian metric g on M, locally given by:

$$g\left(\frac{\delta}{\delta x^{\alpha}}, \frac{\delta}{\delta x^{\beta}}\right) = h_{\alpha\beta}, \quad g\left(\frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial x^{i}}\right) = g\left(\frac{\partial}{\partial x^{i}}, \frac{\delta}{\delta x^{\alpha}}\right) = 0,$$

$$g\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) = k_{ij}.$$

Since $h_{\alpha\beta}$ depend only on $\{x^{n+1}, ..., x^{n+p}\}$, by Theorem 3.2 we conclude that g is bundle–like for $\mathcal{F}$. ■

Example 3.7. Let (M, g) and (N, h) be two semi–Riemannian manifolds and $\pi : M \longrightarrow N$ be a submersion of M onto N. Then the set of fibers of π defines a foliation $\mathcal{F}$ whose tangent distribution we denote by $\mathcal{D}$ (see Example 2.1.2). A vector field X on M is called **vertical** (resp. **horizontal**) if $X \in \Gamma(\mathcal{D})$ (resp. $X \in \Gamma(\mathcal{D}^{\perp})$), where $\mathcal{D}^{\perp}$ is the complementary orthogonal distribution to $\mathcal{D}$ in TM with respect to g. If the fibers $\pi^{-1}(x)$, $x \in N$, are semi–Riemannian submanifolds of M and π_* preserves the lengths of horizontal vector fields, then π is called a **semi–Riemannian submersion** (cf. O'Neill [O83], p. 212). In this case, if $\{(\mathcal{U}, \varphi) : (x^i, x^{\alpha})\}$ is a foliated chart on M we have $\pi_*\left(\frac{\delta}{\delta x^{\alpha}}\right) = \frac{\partial}{\partial x^{\alpha}}$ and therefore

$$g_{\alpha\beta} = g\left(\frac{\delta}{\delta x^{\alpha}}, \frac{\delta}{\delta x^{\beta}}\right) = h\left(\frac{\partial}{\partial x^{\alpha}}, \frac{\partial}{\partial x^{\beta}}\right) = h_{\alpha\beta}(x^{n+1}, ..., x^{n+p}).$$

Thus by Theorem 3.2 g is bundle–like for $\mathcal{F}$. ■

More examples of foliations with bundle–like metric arise as level sets of mappings or as orbits of group actions (see Reinhart [Rei83], pp. 161–163).

As we defined a bundle–like metric on a foliated semi–Riemannian manifold by a condition on the intrinsic connection $D^{\perp}$ on $\mathcal{D}^{\perp}$, we expect that this connection, in this case, must have some special properties. We give some of these properties in the next theorem.

Theorem 3.5. *Let $(M, g, \mathcal{F})$ be a foliated semi–Riemannian manifold as in Theorem* 3.4. *Then we have the following assertions:*

(i) *The local coefficients of the intrinsic connection $D^{\perp}$ on $\mathcal{D}^{\perp}$ with respect to $\left\{\frac{\partial}{\partial x^{i}}, \frac{\delta}{\delta x^{\alpha}}\right\}$ are given by*

$$\text{(a) } L_{\alpha}{}^{\beta}{}_{i} = 0, \quad \text{(b) } F_{\alpha}{}^{\beta}{}_{\gamma} = \frac{1}{2}\, g^{\beta\mu}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\gamma}} + \frac{\partial g_{\mu\gamma}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\gamma}}{\partial x^{\mu}}\right). \qquad (3.16)$$

(ii) *The local components of the curvature tensor field R' of $D^{\perp}$ are given by*

$$\begin{aligned} &\text{(a) } R'{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma} = \frac{\partial F_{\alpha}{}^{\varepsilon}{}_{\beta}}{\partial x^{\gamma}} - \frac{\partial F_{\alpha}{}^{\varepsilon}{}_{\gamma}}{\partial x^{\beta}} + F_{\alpha}{}^{\mu}{}_{\beta}\, F_{\mu}{}^{\varepsilon}{}_{\gamma} - F_{\alpha}{}^{\mu}{}_{\gamma}\, F_{\mu}{}^{\varepsilon}{}_{\beta}, \\ &\text{(b) } R'{}_{\alpha}{}^{\varepsilon}{}_{\beta i} = 0, \\ &\text{(c) } R'{}_{\alpha}{}^{\varepsilon}{}_{ij} = 0. \end{aligned} \tag{3.17}$$

(iii) *The transversal Bianchi identities for the Vrănceanu connection are given by*

$$\begin{aligned} &\text{(a) } \sum_{(\alpha,\beta,\gamma)} \{R'{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\} = 0, \\ &\text{(b) } \sum_{(\alpha,\beta,\gamma)} \{R'{}_{\mu}{}^{\varepsilon}{}_{\alpha\beta|\gamma}\} = 0, \\ &\text{(c) } R'{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma\|j} = \frac{\partial R'{}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}}{\partial x^{j}} = 0. \end{aligned} \tag{3.18}$$

Proof. First, (3.16) follows from (1.26) by using (3.5). Then taking into account that $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are functions of (x^{γ}) alone, from (3.16b) we deduce that $F_{\alpha}{}^{\beta}{}_{\gamma}$ are so. Thus (3.17) follows from (1.31), (1.32) and (1.33). Finally, we use (3.17b) and (3.17c) in (1.61)–(1.65) and obtain (3.18). ∎

Next, we need (3.18) expressed in an invariant form. As the Vrănceanu connection ∇^* on $(M, g, \mathcal{F})$ is an adapted connection on M, from (2.3.27) we deduce that

$$R^*(X, Y)Q'Z = R'(X, Y)Q'Z, \quad \forall\, X, Y, Z \in \Gamma(TM), \tag{3.19}$$

where R^* and R' are the curvature tensors of ∇^* and $D^{\perp}$ respectively. Taking into account that g is bundle–like, that is, g is Vrănceanu–parallel on $\mathcal{D}^{\perp}$, we can apply results from Section 1.7, but for the transversal distribution $\mathcal{D}^{\perp}$. First we put

$$\begin{aligned} R'(Q'U, Q'Z, Q'X, Q'Y) = g(R'(Q'X, Q'Y)Q'Z, Q'U), \\ \forall\, X, Y, Z, U \in \Gamma(TM). \end{aligned} \tag{3.20}$$

It is easy to see that R' defined by (3.20) is a transversal tensor field of type $(0, 0; 0, 4)$ (see Section 2.3). Its main properties are given next.

Theorem 3.6. *Let $(M, g, \mathcal{F})$ be as in Theorem 3.4. Then the curvature tensor field R' of the intrinsic connection $D^{\perp}$ on $\mathcal{D}^{\perp}$ satisfies the following identities:*

$$\begin{aligned} &\text{(a) } R'(Q'U, Q'Z, Q'X, Q'Y) + R'(Q'U, Q'Z, Q'Y, Q'X) = 0, \\ &\text{(b) } R'(Q'U, Q'Z, Q'X.Q'Y) + R'(Q'Z, Q'U, Q'X, Q'Y) = 0, \\ &\text{(c) } R'(Q'U, Q'Z, Q'X, Q'Y) = R'(Q'X, Q'Y, Q'U, Q'Z), \\ &\text{(d) } \sum_{(Q'Z, Q'X, Q'Y)} \{R'(Q'U, Q'Z, Q'X, Q'Y)\} = 0, \end{aligned} \tag{3.21}$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Proof. First, we suppose that $\mathcal{D}^{\perp}$ is integrable. Then, by using (1.6.29), (3.19), (3.14), (1.6.13) and (1.6.14) we deduce that

$$\begin{aligned} R'(Q'U, Q'Z, Q'X, Q'Y) &= g(\widetilde{R}(Q'X, Q'Y)Q'Z, Q'U) \\ &= \widetilde{R}(Q'U, Q'Z, Q'X, Q'Y), \end{aligned} \tag{3.22}$$

where $\widetilde{R}$ is the curvature tensor field of the Levi–Civita connection $\widetilde{\nabla}$ on (M, g). As $\widetilde{R}$ satisfies all identities (3.21) (cf. O'Neill [O83], p.75), from (3.22) we conclude that they are also satisfied by R'. In case $\mathcal{D}^{\perp}$ is not integrable we apply Lemma 1.7.1 and Corollary 1.7.2 for $\mathcal{D}^{\perp}$, and by using (3.19) we obtain (3.21). ∎

Theorem 3.7. *Let* $(M, g, \mathcal{F})$ *be as in Theorem* 3.4. *Then we have*

$$\sum_{(Q'X, Q'Y, Q'Z)} \left\{ \left(D^{\perp}_{Q'X} R'\right)(Q'Y, Q'Z) \right\}(Q'U) = 0, \;\; \forall\, X, Y, Z \in \Gamma(TM). \tag{3.23}$$

Proof. By using (3.12) and (2.6) we deduce that $T^*(X, Y) \in \Gamma(\mathcal{D})$, for any $X, Y \in \Gamma(TM)$. Then by (3.19) and (3.17b) we infer that

$$R^*(T^*(Q'X, Q'Y), Q'Z)Q'U = R'(T^*(Q'X, Q'Y), Q'Z)Q'U = 0. \tag{3.24}$$

Finally, by using (3.24) in (2.4.19) and taking into account that

$$\nabla^*_X Q'Y = D^{\perp}_X Q'Y, \;\; \forall\, X, Y \in \Gamma(TM), \tag{3.25}$$

we obtain (3.23). ∎

Remark 3.8. Clearly, (3.21d) and (3.23) represent the coordinate–free form of (3.18a) and (3.18b) respectively. However, we presented here new proofs based on the geometry of distributions developed so far. ∎

In case $\mathcal{D}^{\perp}$ is integrable and g is bundle–like for $\mathcal{F}$, from (1.6.5) we deduce that

$$\widetilde{R}(Q'U, Q'Z, Q'X, Q'Y) = g(R^{\perp}(Q'X, Q'Y)Q'Z, Q'U), \tag{3.26}$$

where $R^{\perp}$ is the curvature tensor field of the induced connection on $\mathcal{D}^{\perp}$. Thus the sectional curvature of any leaf of $\mathcal{D}^{\perp}$ is just the restriction of the sectional curvature of (M, g) to non–degenerate planes lying in $\mathcal{D}^{\perp}$. This is not surprising because by (3.14) any leaf of $\mathcal{D}^{\perp}$ is totally geodesic immersed in (M, g).

Next we consider the case when $\mathcal{D}^{\perp}$ is not integrable but g is bundle–like for $\mathcal{F}$. Thus $(M, g, \mathcal{F})$ is a foliated semi–Riemannian manifold, where $\mathcal{F}$ is a non–degenerate foliation and g is bundle–like for $\mathcal{F}$, but $(M, g, \mathcal{D}^{\perp})$ is a semi–Riemannian non–holonomic manifold (see the terminology in Section 1.7). Then according to (1.7.14) and (3.19), we can define the **Vrănceanu sectional curvature** of $\mathcal{D}^{\perp}$ as a real–valued function K' on the set of all non–degenerate planes in $\mathcal{D}^{\perp}$, given by

$$K'(Q'X \wedge Q'Y) = \frac{R'(Q'X, Q'Y, Q'X, Q'Y)}{\Delta(Q'X, Q'Y)}, \tag{3.27}$$

where at any point $x \in M$, $\{Q'X, Q'Y\}$ represents a basis of a non–degenerate plane in $\mathcal{D}^{\perp}_x$. When K' does not depend on the non–degenerate planes in $\mathcal{D}^{\perp}$ we say that $\mathcal{D}^{\perp}$ is of **scalar Vrănceanu sectional curvature** K'.

Now, we are able to state a theorem which is a generalization of Schur Theorem from Riemannian (semi–Riemannian) geometry.

Theorem 3.8. *Let $(M, g, \mathcal{F})$ be a foliated connected semi–Riemannian manifold, where $\mathcal{F}$ is a non–degenerate foliation and g is bundle–like for $\mathcal{F}$. Suppose that the transversal distribution $\mathcal{D}^{\perp}$ is of scalar Vrănceanu sectional curvature K'. Then K' is a constant, provided $\mathcal{D}^{\perp}$ is a p-distribution with $p > 2$.*

Proof. First we note that K' depends on (x^{α}) alone. This is a consequence of (3.27), taking into account that both $R_{\alpha}{}^{\mu}{}_{\beta\gamma}$ and $g_{\alpha\beta}$ depend on (x^{α}) alone. Then we consider the 4–linear mapping

$$\begin{aligned} &F : (\Gamma(\mathcal{D}^{\perp}))^4 \longrightarrow F(M);\ F(Q'U, Q'Z, Q'X, Q'Y) \\ &= K'(x^{\alpha})(g(Q'U, Q'X)g(Q'Z, Q'Y) - g(Q'Z, Q'X)g(Q'U, Q'Y)). \end{aligned}$$

It is easy to check that F satisfies the same identities (3.21) which were stated for R'. Thus F is a $\mathcal{D}^{\perp}$–curvature–like mapping satisfying

$$K'(x^{\alpha}) = \frac{F(Q'X, Q'Y, Q'X, Q'Y)}{\Delta(Q'X, Q'Y)}.$$

Hence, by Corollary 1.7.4 we deduce that

$$\begin{aligned} R'(Q'U, Q'Z, Q'X, Q'Y) = &\ K'(x^{\alpha})(g(Q'U, Q'X)g(Q'Z, Q'Y) \\ &- g(Q'Z, Q'X)g(Q'U, Q'Y)), \end{aligned}$$

which is equivalent to

$$R'(Q'X, Q'Y)Q'Z = K'(x^{\alpha})(g(Q'Z, Q'Y)Q'X - g(Q'Z, Q'X)Q'Y). \tag{3.28}$$

Taking into account that g is parallel with respect to the intrinsic connection $D^{\perp}$ (cf. (3.1)), from (3.28) we obtain

$$\begin{aligned} &(D^{\perp}_{Q'U} R')(Q'X, Q'Y)Q'Z \\ &= Q'U(K'(x^{\alpha}))(g(Q'Z, Q'Y)Q'X - g(Q'Z, Q'X)Q'Y). \end{aligned}$$

Then by using (3.23) we infer that

$$\begin{aligned} 0 = & \; Q'U(K'(x^\alpha))(g(Q'Z,Q'Y)Q'X - g(Q'Z,Q'X)Q'Y) \\ & + Q'X(K'(x^\alpha))(g(Q'Z,Q'U)Q'Y - g(Q'Z,Q'Y)Q'U) \\ & + Q'Y(K'(x^\alpha))(g(Q'Z,Q'X)Q'U - g(Q'Z,Q'U)Q'X). \end{aligned} \tag{3.29}$$

Since $p > 2$, for an arbitrary $Q'X$, we may choose $Q'Y$ and $Q'Z$ such that $Q'X, Q'Y$ and $Q'Z$ are mutually orthogonal nowhere zero vector fields. Finally, take $Q'U = Q'Z$ and $Q'X = \dfrac{\delta}{\delta x^\alpha}$ in (3.29) and obtain

$$0 = \frac{\delta K'}{\delta x^\alpha} = \frac{\partial K'}{\partial x^\alpha} - A^i_\alpha \, \frac{\partial K'}{\partial x^i} \, .$$

As K' depends on (x^α) alone, we deduce that $\dfrac{\partial K'}{\partial x^\alpha} = 0$, for any $\alpha \in \{n+1, ..., n+p\}$, that is, K' is a constant on M. ∎

Remark 3.9. When $\mathcal{D} = \{0\}$, that is $\mathcal{D}^\perp = TM$, the intrinsic connection on $D^\perp$ is just the Levi–Civita connection on M, and thus Theorem 3.8 becomes the well known Schur Theorem on (M, g). ∎

If K' is a constant on M then we say that $(M, g, \mathcal{F})$ is a foliated manifold of **constant transversal Vrănceanu curvature**. Then from (3.28) we deduce that the curvature tensor field R' of the intrinsic connection $D^\perp$ satisfies

$$R'(Q'X, Q'Y)Q'Z = c\{g(Q'Z, Q'Y)Q'X - g(Q'Z, Q'X)Q'Y\}, \tag{3.30}$$

for any $X, Y, Z \in \Gamma(TM)$, provided $(M, g, \mathcal{F})$ is of constant transversal Vrănceanu curvature c. By using a general result about Vrănceanu curvature of distributions (see Corollary 1.7.8) we may state the following interesting result.

Theorem 3.9. *Let M be an open submanifold of the Euclidean space $(\mathbb{R}^m, g)$ and $(M, g, \mathcal{F})$ be a foliated Riemannian manifold such that g is bundle–like and $(M, g, \mathcal{D}^\perp)$ is a Riemannian non–holonomic manifold. Then we have the assertions:*

(i) *At any point of M the Vrănceanu sectional curvature of $\mathcal{D}^\perp$ must be non–negative.*

(ii) *If $(M, g, \mathcal{F})$ is of constant transversal Vrănceanu curvature c, then $c > 0$.*

The example presented at the end of Section 1.7 proves the existence of foliated Riemannian manifolds of positive constant transversal Vrănceanu curvature. Thus the problem of classifying foliated Riemannian (semi–Riemannian) manifolds of constant transversal Vrănceanu curvature is a natural, interesting and non-easy open problem that deserves to be addressed.

Now we are in a position to introduce the transversal Ricci tensor and transversal scalar curvature of a semi–Riemannian foliated manifold $(M, g, \mathcal{F})$. To this end we consider an orthonormal frame field $\{E_\alpha\}$, $\alpha \in \{n+1, ..., n+p\}$ for the transversal distribution $\mathcal{D}^\perp$, and denote by $\{\varepsilon_\alpha\}$ the **signature** of $\{E_\alpha\}$, that is $\varepsilon_\alpha = g(E_\alpha, E_\alpha)$. Then we define the **transversal Ricci tensor** $\mathrm{Ric}^{\mathrm{tr}}$ by (see (3.20))

$$\mathrm{Ric}^{\mathrm{tr}}(Q'X, Q'Y) = \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha R'(E_\alpha, Q'Y, E_\alpha, Q'X). \tag{3.31}$$

It is easy to check that $\mathrm{Ric}^{\mathrm{tr}}$ is independent of the choice of the orthonormal frame field. Moreover, when g is bundle–like for $\mathcal{F}$, by using (3.21c) we deduce that $\mathrm{Ric}^{\mathrm{tr}}$ is a symmetric adapted tensor field on M of type $(0,0;0,2)$. Next we consider the non–holonomic frame field $\left\{\dfrac{\delta}{\delta x^\alpha}\right\}$ defined by (1.21) and put

$$\text{(a) } E_\alpha = E_\alpha^\gamma \frac{\delta}{\delta x^\gamma} \quad \text{and} \quad \text{(b) } \frac{\delta}{\delta x^\alpha} = \bar{E}_\alpha^\gamma E_\gamma. \tag{3.32}$$

Then we obtain (see (1.24))

$$\text{(a) } g_{\alpha\beta} = \sum_{\gamma=n+1}^{n+p} \varepsilon_\gamma \bar{E}_\alpha^\gamma \bar{E}_\beta^\gamma \quad \text{and} \quad \text{(b) } g^{\alpha\beta} = \sum_{\gamma=n+1}^{n+p} \varepsilon_\gamma E_\gamma^\alpha E_\gamma^\beta. \tag{3.33}$$

We also put

$$\mathrm{Ric}^{\mathrm{tr}}\left(\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) = R'_{\alpha\beta},$$

and by using (3.31) and (3.33) we deduce that

$$R'_{\alpha\beta} = R'{}_\alpha{}^\gamma{}_{\beta\gamma}. \tag{3.34}$$

Finally, by using (3.31), (3.27) and (1.7.13) we obtain

$$\mathrm{Ric}^{\mathrm{tr}}(E_\gamma, E_\gamma) = \varepsilon_\gamma \sum_{\substack{\alpha=n+1\\ \alpha\neq\gamma}}^{n+p} K'(E_\alpha \wedge E_\gamma), \tag{3.35}$$

for any vector field E_γ from the orthonormal frame field $\{E_\alpha\}$ on $\mathcal{D}^\perp$.

The **transversal scalar curvature** of $(M, g, \mathcal{F})$ is a function on M denoted by S^{tr} and defined by

$$S^{\mathrm{tr}} = \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha \,\mathrm{Ric}^{\mathrm{tr}}(E_\alpha, E_\alpha), \tag{3.36}$$

where $\{E_\alpha\}$ is an orthonormal frame field in $\Gamma(\mathcal{D}^\perp)$. Then by using (3.36) and (3.33b) we obtain

$$S^{\mathrm{tr}} = g^{\alpha\beta} R'_{\alpha\beta}. \tag{3.37}$$

Also, taking into account (3.36), (3.31) and (3.27) we can express the transversal scalar curvature by means of Vrănceanu sectional curvature K' of $\mathcal{D}^\perp$, as follows

$$S^{\mathrm{tr}} = \sum_{\alpha\neq\beta} K'(E_\alpha \wedge E_\beta) = 2\sum_{\alpha<\beta} K'(E_\alpha \wedge E_\beta). \tag{3.38}$$

Theorem 3.10. *Let $(M, g, \mathcal{F})$ be a foliated connected semi–Riemannian manifold, where $\mathcal{F}$ is a non–degenerate foliation and g is bundle–like for $\mathcal{F}$. If $\mathrm{Ric}^{\mathrm{tr}} = \lambda g$, where λ is a smooth function on M, then λ is necessarily a constant provided $\mathcal{D}^\perp$ is a p-distribution with $p > 2$.*

Proof. First we put

$$R'_{\alpha\beta\gamma\mu} = g_{\beta\varepsilon} R'{}_{\alpha}{}^{\varepsilon}{}_{\gamma\mu}. \tag{3.39}$$

Then we see that (3.21a) and (3.21b) imply

$$R'_{\alpha\beta\gamma\mu} = -R'_{\beta\alpha\gamma\mu} = -R'_{\alpha\beta\mu\gamma}. \tag{3.40}$$

Also, by using (3.34), (3.39) and the hypothesis on $\mathrm{Ric}^{\mathrm{tr}}$ we obtain

$$R'_{\alpha\beta} = g^{\gamma\mu} R'_{\alpha\gamma\beta\mu} = \lambda g_{\alpha\beta}. \tag{3.41}$$

Taking into account (3.39), (1.42b) and the Bianchi identities (3.18b) we deduce that

$$R'_{\alpha\beta\gamma\mu|\delta} + R'_{\alpha\beta\mu\delta|\gamma} + R'_{\alpha\beta\delta\gamma|\mu} = 0. \tag{3.42}$$

Contracting (3.42) by $g^{\alpha\gamma} g^{\beta\mu}$ and using (1.42c), (3.40), (1.42b) and (3.41) we obtain

$$(p-2)\lambda_{|\delta} = 0.$$

As $p > 2$, we conclude that $\lambda_{|\delta} = 0$, that is (see (1.21))

$$0 = \lambda_{|\alpha} = \frac{\delta\lambda}{\delta x^\alpha} = \frac{\partial\lambda}{\partial x^\alpha} - A^i_\alpha \,\frac{\partial\lambda}{\partial x^i}\,.$$

Finally, from (3.41) we deduce that λ is a function of (x^α) alone, and thus $\dfrac{\partial\lambda}{\partial x^i} = 0$, for all $i \in \{1, ..., n\}$. Hence $\dfrac{\partial\lambda}{\partial x^\alpha} = 0$, for all $\alpha \in \{n+1, ..., n+p\}$. As M is connected, we deduce that λ is a constant on M. ∎

According to the terminology from Riemannian (semi–Riemannian) geometry, we call $(M, g, \mathcal{F})$ a **transversal Einstein foliated semi–Riemannian manifold**, if the transversal Ricci tensor satisfies

$$\mathrm{Ric}^{\mathrm{tr}} = \lambda g, \tag{3.43}$$

where λ is a constant on M. By using (3.43) and (3.36) we deduce that $\lambda = S^{\mathrm{tr}}/p$, and therefore (3.43) becomes

$$\mathrm{Ric}^{\mathrm{tr}} = \frac{S^{\mathrm{tr}}}{p}\, g \quad \text{or} \quad R'_{\alpha\beta} = \frac{S^{\mathrm{tr}}}{p}\, g_{\alpha\beta}. \tag{3.44}$$

Theorem 3.11. *Let $(M, g, \mathcal{F})$ be a foliated connected semi–Riemannian manifold, where $\mathcal{F}$ is a non–degenerate foliation and g is bundle–like for $\mathcal{F}$. If $(M, g, \mathcal{F})$ is of constant transversal Vrănceanu curvature then it is transversal Einstein.*

Proof. Let $\{E_\alpha\}$ be an orthonormal basis in $\Gamma(\mathcal{D}^\perp)$. Then any $Q'X \in \Gamma(\mathcal{D}^\perp)$ is expressed as follows (see Lemma 1.4.1)

$$Q'X = \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha g(Q'X, E_\alpha)E_\alpha, \ \varepsilon_\alpha = g(E_\alpha, E_\alpha). \tag{3.45}$$

This implies

$$g(Q'X, Q'Y) = \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha g(Q'X, E_\alpha)g(Q'Y, E_\alpha), \tag{3.46}$$

for any $Q'X, Q'Y \in \Gamma(\mathcal{D}^\perp)$. Now, by using (3.31), (3.20), (3.30) and (3.46) we obtain

$$\begin{aligned} \mathrm{Ric}^{\mathrm{tr}}(Q'X, Q'Y) &= c \sum_{\alpha=n+1}^{n+p} \{g(Q'X, Q'Y) - \varepsilon_\alpha g(Q'X, E_\alpha)g(Q'Y, E_\alpha)\} \\ &= c(p-1)g(Q'X, Q'Y). \end{aligned} \tag{3.47}$$

Thus $(M, g, \mathcal{F})$ is transversal Einstein. ∎

The next theorem is a generalization of a result obtained by Schouten and Struik [SS21].

Theorem 3.12. *Let $(M, g, \mathcal{F})$ be an $(n+3)$–dimensional transversal Einstein semi–Riemannian foliated manifold, where $\mathcal{F}$ is a non–degenerate n–foliation and g is bundle–like for $\mathcal{F}$. Then $(M, g, \mathcal{F})$ is of constant transversal Vrănceanu curvature.*

Proof. Let $\{E_1, E_2, E_3\}$ be an orthonormal frame field in $\Gamma(\mathcal{D}^{\perp})$. Then by using (3.35) and (3.43) we calculate $\mathrm{Ric}^{\mathrm{tr}}(E_t, E_t)$, $t \in \{1, 2, 3\}$, and obtain

$$\begin{aligned} K'(E_1 \wedge E_2) + K'(E_1 \wedge E_3) &= K'(E_1 \wedge E_2) + K'(E_2 \wedge E_3) \\ &= K'(E_1 \wedge E_3) + K'(E_2 \wedge E_3) = \lambda. \end{aligned}$$

Thus we have $K'(E_1 \wedge E_2) = K'(E_1 \wedge E_3) = K'(E_2 \wedge E_3) = \frac{\lambda}{2}$, which completes the proof of the theorem. ∎

We end this section by describing a geometrically interesting class of foliations with bundle–like metric. This is the class of transversally symmetric foliations introduced by Tondeur and Vanhecke [TV90]. Roughly speaking, these are Riemannian foliations whose transversal geometry is locally modelled on a Riemannian symmetric space. To be more specific we proceed as follows.

Let $\mathcal{F}$ be an n–foliation of an $(n+p)$–dimensional manifold M and $\{(\mathcal{U}, \varphi) : (x^i, x^\alpha)\}$ be a foliated chart on M. Taking into account that any submersion is an open mapping (cf. Brickell–Clark [BC70], p. 87) and by using Remark 2.1.3 we conclude that the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a submersion $\pi : \mathcal{U} \longrightarrow \mathcal{V} \subset \mathbb{R}^p$ onto an open subset $\mathcal{V}$ of $\mathbb{R}^p$. Next, we suppose that g is a bundle–like Riemannian metric on M for the foliation $\mathcal{F}$. Then according to (3.5), the functions $g_{\alpha\beta}$ given by (1.24) define a Riemannian metric on $\mathcal{V}$. Since the induced Riemannian metric by g on $\mathcal{D}^{\perp}_{|\mathcal{U}}$ is also given by $g_{\alpha\beta}$, we may claim that $\pi : \mathcal{U} \longrightarrow \mathcal{V}$ is a Riemannian submersion. Hence the plaques of $\mathcal{F}$ in $\mathcal{U}$ are the fibers of a Riemannian submersion $\pi : \mathcal{U} \longrightarrow \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a transversal model Riemannian manifold N. This justifies the name Riemannian foliation for $\mathcal{F}$. Then following Tondeur–Vanhecke [TV90] we say that the foliation $\mathcal{F}$ with bundle–like metric g is **transversally symmetric** if N is a locally symmetric Riemannian space. To be more specific, we take a point $x \in N$ and a normal neighbourhood $\mathcal{V}_x$ of x. Then for each $y \in \mathcal{V}_x$ consider the geodesic $t \longrightarrow \gamma(t)$ within $\mathcal{V}_x$ passing through x and y such that $\gamma(0) = x$ and $\gamma(1) = y$. The mapping $y \longrightarrow y' = \gamma(-1)$ of $\mathcal{V}_x$ onto itself is called the **geodesic symmetry** (**reflection**) with respect to x. Now, according to Helgason [Hel01], p. 200, N is called a **locally symmetric Riemannian space** if for each $x \in N$ there exists a normal neighbourhood of x on which the geodesic symmetry with respect to x is an isometry.

Next, to state some characterizations of transversally symmetric foliations we need the following. We considered in Section 1.6 the tensor field A (see (1.6.33)) which was introduced by O'Neill [O66] for submersions. In case of a foliation with bundle–like metric, by using (1.6.33) and (3.10) we deduce that

$$A_X Y = \frac{1}{2} Q[Q'X, Q'Y], \quad \forall\, X, Y \in \Gamma(TM). \tag{3.48}$$

Remark 3.10. By using (3.10), (3.12) and (3.48) we deduce that in case $\mathcal{F}$ is a foliation with bundle–like metric, the torsion tensor field T^* of the Vrănceanu connection ∇^* and the O'Neill tensor field A are related by

$$T^* = -2A. \tag{3.49}$$

Also, from (3.48) and (3.49) it follows that both T^* and A are obstructions to the integrability of the distribution $\mathcal{D}^\perp$. ■

Finally, we denote by $\widetilde{R}$ and R^* the curvature tensor fields of the Levi–Civita connection $\widetilde{\nabla}$ on (M, g) and of the Vrănceanu connection ∇^* defined by $\widetilde{\nabla}$. Then we put

$$\begin{aligned} &\text{(a) } \widetilde{R}(X, Y, Z, U) = g(\widetilde{R}(Z, U)Y, X), \\ &\text{(b) } R^*(X, Y, Z, U) = g(R^*(Z, U)Y, X), \ \forall\, X, Y, Z, U \in \Gamma(TM). \end{aligned} \tag{3.50}$$

Taking into account the above discussion we can restate a result due to Tondeur and Vanhecke [TV90] as follows.

Theorem 3.13. *Let $\mathcal{F}$ be a Riemannian foliation on (M, g) and g a bundle–like metric for $\mathcal{F}$. Then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is transversally symmetric.*
(ii) *The local geodesic symmetries on the model space are isometries.*
(iii) $\nabla^*_U(R^*(U, V, U, V)) = 0, \quad \forall\, U, V \in \Gamma(\mathcal{D}^\perp)$.
(iv) $\widetilde{\nabla}_U(\widetilde{R}(U, V, U, V)) + 2\widetilde{R}(U, A_U V, U, V) = -6g((\widetilde{\nabla}_U A)_U V, A_U V)$, $\forall\, U, V \in \Gamma(\mathcal{D}^\perp)$.
(v) $\widetilde{\nabla}_U(\widetilde{R}(U, V, U, V)) - \widetilde{R}(U, T^*(U, V), U, V) = 3g((\widetilde{\nabla}_U T^*)(U, V), T^*(U, V))$, $\forall\, U, V \in \Gamma(\mathcal{D}^\perp)$.

We note that the last three conditions are automatically satisfied when $\mathcal{F}$ is of codimension one. Therefore we have the following corollary.

Corollary 3.14. *Any Riemannian foliation of codimension one is transversally symmetric.*

The geometry of the ambient space M has a strong effect on the existence of transversally symmetric foliations. As an example we give the following.

Corollary 3.15. (Tondeur–Vanhecke [TV90]). *Let $\mathcal{F}$ be a foliation on a space of constant curvature (M, g) such that g is bundle–like for $\mathcal{F}$. If $\mathcal{D}^\perp$ is integrable, then $\mathcal{F}$ is transversally symmetric.*

Results on the influence of the existence of transversally symmetric foliations on the geometry of the ambient manifold can be found in Tondeur–Vanhecke [TV90].

We close this section with a new characterization of transversally symmetric foliations. To state this we consider the semi–holonomic frame field $\left\{\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha}\right\}$ on (M, g). Then by using (3.50) and (3.19) we obtain

$$\begin{aligned} & R^*\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right) \\ & = g\left(R'\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right)\frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha}\right) \qquad (3.51) \\ & = R'_\beta{}^\gamma{}_{\beta\alpha}\, g_{\gamma\alpha}. \end{aligned}$$

Clearly, $\Lambda_{\alpha\beta} = R'_\beta{}^\gamma{}_{\beta\alpha} g_{\gamma\alpha}$ define the local components of an adapted tensor field Λ of type $(0,0;0,2)$ (see Section 2.2). Now we state the following.

Theorem 3.16. *Let $\mathcal{F}$ be a Riemannian n–foliation on an $(n+p)$–dimensional Riemannian manifold (M, g) and g a bundle–like metric for $\mathcal{F}$. Then $\mathcal{F}$ is transversally symmetric if and only if for any fixed pair (α, β), $\alpha, \beta \in \{n+1, ..., n+p\}$, the local components $\Lambda_{\alpha\beta}$ of Λ depend only on x^ε, where $\varepsilon \neq \alpha$ and $\varepsilon \neq \beta$.*

Proof. We take $U = \frac{\delta}{\delta x^\alpha}$ and $V = \frac{\delta}{\delta x^\beta}$ into (iii) of Theorem 3.13 and by using (3.51) we obtain

$$\frac{\delta}{\delta x^\alpha}(\Lambda_{\alpha\beta}) = 0. \qquad (3.52)$$

Since g is bundle–like for $\mathcal{F}$, $g_{\alpha\beta}$ do not depend on x^i, $i \in \{1, ..., n\}$ (see (3.5)). Hence by (3.17a), $R'_\alpha{}^\varepsilon{}_{\beta\gamma}$ do not depend on x^i, and therefore

$$\frac{\partial \Lambda_{\alpha\beta}}{\partial x^i} = 0, \quad \forall\, i \in \{1, ..., n\}. \qquad (3.53)$$

Then, by using (1.21), (3.52) and (3.53), we deduce that

$$\frac{\partial \Lambda_{\alpha\beta}}{\partial x^\alpha} = 0. \qquad (3.54)$$

As $\Lambda_{\alpha\beta} = -\Lambda_{\beta\alpha}$, from (3.53) and (3.54) we conclude that for any fixed pair (α, β), $\Lambda_{\alpha\beta}$ does not depend on (x^i, x^α, x^β), $i \in \{1, ..., n\}$. Thus the proof is complete. ∎

In particular, if $\mathcal{F}$ is of codimension two then Λ has the components $\Lambda_{12} = -\Lambda_{21}$. Thus from Theorem 3.16 we obtain the following.

Corollary 3.17. *Let $\mathcal{F}$ be a Riemannian foliation of codimension two on (M, g) and g a bundle–like metric for $\mathcal{F}$. Then $\mathcal{F}$ is transversally symmetric if and only if for each point $x \in M$ there exists a foliated chart $(\mathcal{U}, \varphi)$ such that Λ_{12} is a constant on $\mathcal{U}$.*

We will visit transversally symmetric foliations again in the next section.

3.4 Special Classes of Foliations

The purpose of this section is to present the main problems related to three important classes of foliations: totally geodesic, totally umbilical and minimal (harmonic) foliations. First, by using both the induced and intrinsic connections on the structural distribution we present several characterizations of totally geodesic foliations. Then we deduce two differential equations of Riccati type and use them for studying the integrability of the transversal distribution and the existence of totally geodesic foliations. By using results from Walschap [Was97] and our theory on structural and transversal differentiations, we give complete characterizations of totally umbilical foliations with bundle–like metric on Riemannian spaces of constant curvature. Finally, by using the intrinsic covariant derivative $D^{\perp}$ and the $D^{\perp}$–divergence operator we introduce and study minimal foliations. Most of the results of the section are presented in the general framework of non–degenerate foliations on semi–Riemannian manifolds.

3.4.1 Totally Geodesic Foliations on Semi–Riemannian Manifolds

Throughout this section $\mathcal{F}$ represents a non–degenerate n–foliation on an $(n+p)$–dimensional semi–Riemannian manifold (M, g). If each leaf of $\mathcal{F}$ is a totally geodesic submanifold of (M, g) we say that $\mathcal{F}$ is a **totally geodesic foliation**. Then by using a well known characterization of totally geodesic submanifolds (cf. O'Neill [O83], p.104) and (2.12c) we can state the following.

Theorem 4.1. *$\mathcal{F}$ is totally geodesic if and only if one of the following conditions is satisfied:*

(i) *The second fundamental form of $\mathcal{F}$ vanishes identically on M, i.e., we have*

$$h(QX, QY) = Q'\widetilde{\nabla}_{QX}QY = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{4.1}$$

(ii) *The shape operator of the structural distribution $\mathcal{D}$ vanishes identically on M, i.e., we have*

$$A_{Q'X}QY = -Q\widetilde{\nabla}_{QY}Q'X = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{4.2}$$

Now, we remark that the symmetric second fundamental form h^s of $\mathcal{D}$ (see (1.5.34)) coincides with h. Then taking into account that $\mathcal{D}$ is integrable, from Theorems 1.5.6, 1.5.9 and 1.5.10 we deduce several characterizations of totally geodesic foliations as follows.

Theorem 4.2. *Let (M, g) be a semi–Riemannian manifold and $\mathcal{F}$ be a non–degenerate foliation on M. Then the following assertions are equivalent:*

(i) *$\mathcal{F}$ is a totally geodesic foliation.*
(ii) *The induced connection ∇ coincides with the intrinsic connection D on $\mathcal{D}$.*
(iii) *g is parallel with respect to the intrinsic connection D on $\mathcal{D}$.*
(iv) *$Q'X$ is a $\mathcal{D}$–Killing vector field, for any $X \in \Gamma(TM)$.*
(v) *The induced connection ∇ on $\mathcal{D}$ is torsion–free.*

Remark 4.1. The condition (iii) was given as characterization of totally geodesic foliations by Sanini [San82]. In our terminology from Section 1.5 this condition can be read as follows:

(iii′) *g is bundle–like for the transversal distribution.* ■

Next, we consider the curvature tensor field $\widetilde{R}$ of the Levi–Civita connection $\widetilde{\nabla}$ on (M, g), the intrinsic connection $D^{\perp}$ on $\mathcal{D}^{\perp}$ and the shape operator A'_{QX} of $\mathcal{D}^{\perp}$ for $X \in \Gamma(TM)$. Then we prove the following.

Lemma 4.3. *Let $\mathcal{F}$ be a totally geodesic foliation on a semi–Riemannian manifold (M, g). Then we have*

$$\begin{aligned} &(D^{\perp}_{QX} A'_{QX})(Q'Y) = (A'_{QX})^2(Q'Y) \\ &-Q'\widetilde{R}(QX, Q'Y)QX + A'_{\nabla_{QX}QX} Q'Y, \end{aligned} \tag{4.3}$$

for any $X, Y \in \Gamma(TM)$.

Proof. By direct calculations using (1.9), (1.5.8), (2.11) and (4.1) we obtain

$$\begin{aligned} &(D^{\perp}_{QX} A'_{QX})(Q'Y) = D^{\perp}_{QX}(A'_{QX}Q'Y) - A'_{QX}(D^{\perp}_{QX}Q'Y) \\ &= Q'[QX, A'_{QX}Q'Y] - A'_{QX}(Q'[QX, Q'Y]) \\ &= Q'(\widetilde{\nabla}_{QX}(A'_{QX}Q'Y)) - Q'(\widetilde{\nabla}_{A'_{QX}Q'Y}QX) + Q'\widetilde{\nabla}_{Q'[QX,Q'Y]}QX \\ &= A'_{QX}(A'_{QX}Q'Y) + Q'(\widetilde{\nabla}_{[QX,Q'Y]}QX - \widetilde{\nabla}_{QX}\widetilde{\nabla}_{Q'Y}QX). \end{aligned} \tag{4.4}$$

On the other hand, by using (1.2.17) and (2.11) we deduce that

$$\begin{aligned} &Q'(\widetilde{\nabla}_{[QX,Q'Y]}QX - \widetilde{\nabla}_{QX}\widetilde{\nabla}_{Q'Y}QX) \\ &= Q'(\widetilde{\nabla}_{[QX,Q'Y]}QX - \widetilde{\nabla}_{QX}\widetilde{\nabla}_{Q'Y}QX + \widetilde{\nabla}_{Q'Y}\widetilde{\nabla}_{QX}QX) \\ &-Q'\widetilde{\nabla}_{Q'Y}\widetilde{\nabla}_{QX}QX = -Q'\widetilde{R}(QX, Q'Y)QX + A'_{\nabla_{QX}QX}Q'Y, \end{aligned} \tag{4.5}$$

since $\widetilde{\nabla}_{QX}QX = \nabla_{QX}QX$. Finally, by using (4.5) in (4.4) we obtain (4.3). ■

Next, by using (2.7d) and (2.11) we deduce that the induced and intrinsic connections $\nabla^{\perp}$ and $D^{\perp}$ on $\mathcal{D}^{\perp}$ are related by

$$\nabla^{\perp}_{QX}Q'Y = D^{\perp}_{QX}Q'Y - A'_{QX}Q'Y, \quad \forall X, Y \in \Gamma(TM). \tag{4.6}$$

Then we prove the following.

Lemma 4.4. *Let $(M, g, \mathcal{F})$ be a foliated semi–Riemannian manifold. Then we have*

$$\nabla^{\perp}_{QX}A'_{QX} = D^{\perp}_{QX}A'_{QX}, \quad \forall X \in \Gamma(TM). \tag{4.7}$$

Proof. By using (4.6) we obtain

$$\begin{aligned}
&(\nabla^{\perp}_{QX}A'_{QX})(Q'Y) = \nabla^{\perp}_{QX}(A'_{QX}Q'Y) - A'_{QX}(\nabla^{\perp}_{QX}Q'Y)\\
&= D^{\perp}_{QX}(A'_{QX}Q'Y) - (A'_{QX})^2QY - A'_{QX}(D^{\perp}_{QX}Q'Y - A'_{QX}Q'Y)\\
&= (D^{\perp}_{QX}A'_{QX})(Q'Y), \text{ for any } Y \in \Gamma(TM),
\end{aligned}$$

which proves (4.7). ∎

Based on (4.7) we can rewrite (4.3) in the equivalent form

$$\begin{aligned}
(\nabla^{\perp}_{QX}A'_{QX})(Q'Y) = {} & (A'_{QX})^2(Q'Y) - Q'\widetilde{R}(QX, Q'Y)QX\\
& + A'_{\nabla_{QX}QX}Q'Y, \ \forall X, Y \in \Gamma(TM).
\end{aligned} \tag{4.8}$$

Now, consider a unit–speed geodesic $\gamma(t)$ that lies in a leaf of the totally geodesic foliation $\mathcal{F}$, that is,

$$\widetilde{\nabla}_{\dot\gamma(t)}\dot\gamma(t) = \nabla_{\dot\gamma(t)}\dot\gamma(t) = 0, \quad g(\dot\gamma(t), \dot\gamma(t)) = 1,$$

where ∇ is the induced connection on $\mathcal{D}$ and $\dot\gamma(t)$ is the tangent vector field to γ. Then by using (4.3) and (4.8) we can state the following.

Theorem 4.5. *Let γ be a unit–speed geodesic that lies in a leaf of a totally geodesic foliation $\mathcal{F}$ on a semi–Riemannian manifold (M, g). Then we have*

$$\begin{aligned}
&(D^{\perp}_{\dot\gamma(t)}A'_{\dot\gamma(t)})(Q'Y) = (\nabla^{\perp}_{\dot\gamma(t)}A'_{\dot\gamma(t)})(Q'Y)\\
&= (A'_{\dot\gamma(t)})^2(Q'Y) - Q'\widetilde{R}(\dot\gamma(t), Q'Y)\dot\gamma(t),
\end{aligned} \tag{4.9}$$

for any $Y \in \Gamma(TM)$.

The equation (4.9) is known as a Riccati type differential equation (cf. K. Abe [Abe73]), and it was first obtained by D. Ferus [Fer70] for totally geodesic foliations on a Riemannian manifold. Also, D. Ferus [Fer70] proved that the dimension of leaves of a totally geodesic foliation on a Riemannian manifold cannot exceed a certain limit, provided the leaves are complete and the sectional curvature of M has the same positive value for all planes spanned by $\{QX, Q'Y\}$, $X, Y \in \Gamma(TM)$. In other words, the codimension of a such totally geodesic foliation is either zero or large.

For the next results we restrict our study to totally geodesic foliations on Riemannian manifolds. In this case we show that some conditions on the sectional curvature of the ambient manifold have a great impact on the transversal geometry of the foliation. To state these results we denote by $\widetilde{K}(X \wedge Y)$ the sectional curvature of (M, g) for the plane determined by $\{X, Y\}$. Then we call $\widetilde{K}(QX \wedge Q'Y)$ the **mixed sectional curvature** determined by $QX \in \Gamma(\mathcal{D})$ and $Q'Y \in \Gamma(\mathcal{D}^{\perp})$. First we prove the following.

Theorem 4.6. *Let $\mathcal{F}$ be a totally geodesic foliation on a Riemannian manifold (M, g). If all mixed sectional curvatures of M at a point x_0 are positive, then the transversal distribution is not integrable.*

Proof. Let u be a non-zero vector in $\mathcal{D}_{x_0}$. Then there exists a vector field $QX \in \Gamma(\mathcal{D})$ on a neighbourhood $\mathcal{U} \subset M$ of x_0 such that $QX(x_0) = u$ and

$$(\nabla_{QX} QX)(x_0) = (\widetilde{\nabla}_{QX} QX)(x_0) = 0, \tag{4.10}$$

where ∇ is the induced connection by $\widetilde{\nabla}$ on $\mathcal{D}$. Now, suppose by absurd that $\mathcal{D}^{\perp}$ is integrable. Then by the assertion (ii) of Corollary 2.4 we deduce that A'_{QX} is self–adjoint. Next, consider A'_{QX} restricted to the local leaf $\mathcal{U}^{\perp} = \mathcal{U} \cap L^{\perp}$, where $L^{\perp}$ is the leaf of $\mathcal{D}^{\perp}$ through x_0. Suppose λ is an eigenfunction of A'_{QX} on $\mathcal{U}^{\perp}$ with unit eigenvector field $Q'Y$. Then by using (4.8), (4.10) and a formula for $\widetilde{K}$ similar to (1.7.14), we obtain

$$\begin{aligned} &g((\nabla^{\perp}_{QX} A'_{QX})(Q'Y), Q'Y)(x_0) \\ &= \lambda^2(x_0) + \widetilde{K}(QX \wedge Q'Y)(x_0)\varDelta(QX, Q'Y)(x_0), \end{aligned} \tag{4.11}$$

where $\varDelta$ is given by (1.7.13). On the other hand, taking into account that g is parallel with respect to the induced connection $\nabla^{\perp}$ (cf. Lemma 1.5.5) and that A'_{QX} is self–adjoint, the left hand side in (4.11) becomes

$$\begin{aligned} &g(\nabla^{\perp}_{QX}(A'_{QX} Q'Y) - A'_{QX}(\nabla^{\perp}_{QX} Q'Y), Q'Y)(x_0) \\ &= g(\nabla^{\perp}_{QX}(\lambda Q'Y), Q'Y)(x_0) - g(\nabla^{\perp}_{QX} Q'Y, \lambda Q'Y)(x_0) = QX(\lambda)(x_0). \end{aligned}$$

As λ is a function of (x^{α}) alone and $QX = X^i \dfrac{\partial}{\partial x^i}$ we deduce that $QX(\lambda)(x_0) = 0$. But both $\widetilde{K}(QX \wedge Q'Y)(x_0)$ and $\varDelta(QX, Q'Y)(x_0)$ are positive, so the right part in (4.11) is a positive number. Thus, from (4.11), we have a contradiction which proves our theorem. ∎

When M is supposed to have positive sectional curvatures, Theorem 4.6 was obtained by K. Abe [Abe73]. Also, we have the following.

Corollary 4.7. *Let (M, g) be a Riemannian manifold whose mixed sectional curvatures at a point x_0 are positive. Then there exist no totally geodesic foliations of codimension 1 on (M, g).*

Proof. If $\mathcal{F}$ is totally geodesic and of codimension 1, then $D^{\perp}$ is a line distribution, so it is integrable. This is impossible by Theorem 4.6. ∎

Corollary 4.8. (K. Abe [Abe73]). *Let (M, g) be a 2–dimensional manifold with positive Gaussian curvature. Then any totally geodesic foliation is a trivial one. In particular, any vector field on M whose integral curves are geodesics must have at least one zero.*

Proof. If $\mathcal{F}$ is not a trivial foliation, then $\mathcal{D}^{\perp}$ is a line distribution, so it is integrable. By Theorem 4.6 this is impossible. Clearly, the second part of the corollary is a consequence of the first part. ∎

Based on our general formula (4.8) we prove the following.

Theorem 4.9. (Tanno [Tan72]). *Let $\mathcal{F}$ be a totally geodesic foliation on a Riemannian manifold (M, g). Suppose that all mixed sectional curvatures of M vanish identically on M and the transversal distribution $\mathcal{D}^{\perp}$ is integrable. Then the foliation $\mathcal{F}^{\perp}$ defined by $\mathcal{D}^{\perp}$ is also totally geodesic.*

Proof. By (2.12d) and Theorem 4.1 for $\mathcal{D}^{\perp}$ we deduce that $\mathcal{F}^{\perp}$ is totally geodesic if and only if

$$A'_{QX} = 0, \quad \forall X \in \Gamma(TM).$$

Suppose by absurd that $\mathcal{F}^{\perp}$ is not totally geodesic. Thus there exist a point $x_0 \in M$ and a vector $u \in \mathcal{D}_{x_0}$ such that A'_u is a non-zero linear operator on $\mathcal{D}^{\perp}_{x_0}$. Then consider a vector field $QX \in \Gamma(\mathcal{D})$ on a neighbourhood $\mathcal{U} \subset M$ of x_0 such that $QX(x_0) = u$ and (4.10) is satisfied. Making $\mathcal{U}$ smaller if necessary, by continuity we may suppose that $A'_{QX} \neq 0$ on $\mathcal{U}$. Now, we take the restriction of A'_{QX} to $\mathcal{U}^{\perp} = \mathcal{U} \cap L^{\perp}$, where $L^{\perp}$ is the leaf of $\mathcal{D}^{\perp}$ through x_0. Since A'_{QX} is a non-zero self-adjoint operator on $\Gamma(\mathcal{D}^{\perp}_{|\mathcal{U}^{\perp}})$ (cf. (ii) of Corollary 2.4) it has a non-zero eigenfunction λ on $\mathcal{U}^{\perp}$. Then we take $Q'Y$ from (4.8) as a unit eigenvector field associated to λ, and by using (4.10) and the hypothesis, that is, $\widetilde{K}(QX \wedge Q'Y) = 0$, we deduce that

$$g((\nabla^{\perp}_{QX} A'_{QX})(Q'Y), Q'Y) = \lambda^2.$$

In a similar way as in the proof of Theorem 4.6 it follows that the left hand side of the above equality vanishes on $\mathcal{U}^{\perp}$. As $\lambda \neq 0$ on $\mathcal{U}^{\perp}$ we get a contradiction. This completes the proof of the theorem. ∎

In particular, we deduce that (M, g) from Theorem 4.9 is locally a Riemannian product of local leaves of $\mathcal{D}$ and $\mathcal{D}^{\perp}$. By using other geometrical conditions on totally geodesic foliations, K. Abe [Abe73], Brito and Walczak [BW86] and R.H. Escobales Jr. [Esc82] obtained such theorems of decomposition of (M, g).

Next, we study the existence of totally geodesic foliations with bundle–like metrics and subject to some curvature conditions.

First, we state a lemma whose proof is similar to that of Lemma 4.4.

Lemma 4.10. *Let ∇ and D be the induced and intrinsic connections on the structural distribution of a non–degenerate foliation $\mathcal{F}$ on (M, g). Then we have*

$$\nabla_{Q'X} A_{Q'X} = D_{Q'X} A_{Q'X}, \quad \forall X \in \Gamma(TM). \tag{4.12}$$

Now we prove the following.

Lemma 4.11. *Let $(M, g, \mathcal{F})$ be a foliated semi–Riemannian manifold, where $\mathcal{F}$ is a non–degenerate foliation and g is bundle–like for $\mathcal{F}$. Then we have*

$$\begin{aligned} (D_{Q'X} A_{Q'X})QY = \ & (\nabla_{Q'X} A_{Q'X})QY = (A_{Q'X})^2 QY \\ & + Q\widetilde{R}(QY, Q'X)Q'X - h'(h(Q'X, QY), Q'X) \\ & - Q\widetilde{\nabla}_{QY} \widetilde{\nabla}_{Q'X} Q'X, \quad \forall X, Y \in \Gamma(TM), \end{aligned} \tag{4.13}$$

where $\widetilde{R}$ is the curvature tensor field of the Levi–Civita connection $\widetilde{\nabla}$ on (M, g), and h and h' are given by (2.4).

Proof. First, by using (2.11) for h, (2.9) and (1.5.8) we obtain

$$\begin{aligned} & (D_{Q'X} A_{Q'X})QY = D_{Q'X}(A_{Q'X} QY) - A_{Q'X}(D_{Q'X} QY) \\ & = Q[Q'X, A_{Q'X} QY] - A_{Q'X}(Q[Q'X, QY]) \\ & = Q\widetilde{\nabla}_{Q'X}(A_{Q'X} QY) - Q\widetilde{\nabla}_{A_{Q'X} QY} Q'X + Q\widetilde{\nabla}_{Q[Q'X, QY]} Q'X \\ & = (A_{Q'X})^2 (QY) + Q(\widetilde{\nabla}_{Q[Q'X, QY]} Q'X - \widetilde{\nabla}_{Q'X} Q\widetilde{\nabla}_{QY} Q'X). \end{aligned}$$

Taking into account that Q and Q' are complementary projectors, and by using (1.2.17) for $\widetilde{R}$ and (2.6) we deduce that

$$\begin{aligned} & (D_{Q'X} A_{Q'X})(QY) = (A_{Q'X})^2 (QY) \\ & + Q(\widetilde{\nabla}_{[Q'X, QY]} Q'X - \widetilde{\nabla}_{Q'X} \widetilde{\nabla}_{QY} Q'X + \widetilde{\nabla}_{QY} \widetilde{\nabla}_{Q'X} Q'X) \\ & - Q(\widetilde{\nabla}_{Q'[Q'X, QY]} Q'X - \widetilde{\nabla}_{Q'X} Q' \widetilde{\nabla}_{QY} Q'X + \widetilde{\nabla}_{QY} \widetilde{\nabla}_{Q'X} Q'X) \\ & = (A_{Q'X})^2 (QY) + Q\widetilde{R}(QY, Q'X)Q'X \\ & - h'(Q'[Q'X, QY], Q'X) + h'(Q'X, Q'\widetilde{\nabla}_{QY} Q'X) \\ & - Q\widetilde{\nabla}_{QY} \widetilde{\nabla}_{Q'X} Q'X. \end{aligned} \tag{4.14}$$

Finally, by using (1.5.8), the assertion (vi) of Theorem 3.3 and (2.11) for h, we infer that

$$\begin{aligned} & h'(Q'X, Q'\widetilde{\nabla}_{QY}Q'X) - h'(Q'[Q'X, QY], Q'X) \\ & = h'(Q'X, Q'\widetilde{\nabla}_{QY}Q'X) + h'(Q'\widetilde{\nabla}_{QY}Q'X, Q'X) \\ & -h'(Q'\widetilde{\nabla}_{Q'X}QY, Q'X) = -h'(h(Q'X, QY), Q'X). \end{aligned} \tag{4.15}$$

Thus by using (4.15) in (4.14) we obtain (4.13). ∎

Next, we consider $\mathcal{F}$ on (M, g) such that g is bundle–like for $\mathcal{F}$. Then we may take a geodesic γ that is tangent to the transversal distribution, that is, $\dot{\gamma}(t) \in \Gamma(\mathcal{D}^{\perp})$ (cf. Remark 3.4). Replace $Q'X$ from (4.13) by $\dot{\gamma}(t)$ and taking into account that $\widetilde{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$, we obtain the Riccati type equation

$$\begin{aligned} D_{\dot{\gamma}(t)}A_{\dot{\gamma}(t)} = \nabla_{\dot{\gamma}(t)}A_{\dot{\gamma}(t)} &= A^2_{\dot{\gamma}(t)} + Q\widetilde{R}(\cdot, \dot{\gamma}(t))\dot{\gamma}(t) \\ & - h'(h(\dot{\gamma}(t), \cdot), \dot{\gamma}(t)). \end{aligned} \tag{4.16}$$

When (M, g) is Riemannian, (4.16) is equivalent to the Riccati type equations obtained by Kim–Tondeur [KT92] and Walschap [Was97].

Theorem 4.12. *Let $\mathcal{F}$ be a totally geodesic non–degenerate n–foliation on an $(n+p)$–dimensional, $p \geq 1$, semi–Riemannian manifold (M, g) such that g is bundle–like for $\mathcal{F}$. If there exists a neighbourhood $\mathcal{U} \subset M$ such that*

$$Q\widetilde{R}(QY, Q'X)Q'X \neq 0, \quad \textit{on } \mathcal{U},$$

for any non-zero vector fields QY and $Q'X$, then $n \leq p - 1$.

Proof. First, for any $Q'X \neq 0$ we consider the $F(M)$–linear operator

$$P_{Q'X} : \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}^{\perp}) : P_{Q'X}(QY) = h(Q'X, QY), \ \forall Y \in \Gamma(TM). \tag{4.17}$$

Then by using (4.17) and (2.12b) we obtain

$$g(P_{Q'X}(QY), Q'X) = -g(h'(Q'X, Q'X), QY) = 0, \tag{4.18}$$

since g is bundle–like for $\mathcal{F}$ (see (3.10)). Now we choose $Q'X$ as a vector field that is not light–like with respect to g at any point of M. Then from (4.18) we deduce that the range of $P_{Q'X}$ lies in the orthogonal complement of span$\{Q'X\}$ in $\Gamma(\mathcal{D}^{\perp})$. Thus we have

$$\operatorname{rank} P_{Q'X} \leq p - 1, \text{ at any point of } M. \tag{4.19}$$

Now, suppose by absurd that $n > p - 1$. Then by (4.17) and (4.19) there exists a non–zero vector field $QY \in \Gamma(\mathcal{D})$ such that $P_{Q'X}(QY) = 0$. Thus for the above choice of both $Q'X$ and QY, (4.13) becomes

$$Q\widetilde{R}(QY, Q'X)Q'X = Q\widetilde{\nabla}_{QY}\widetilde{\nabla}_{Q'X}Q'X, \tag{4.20}$$

since by assertion (ii) of Theorem 4.1, $A_{Q'X}$ vanishes identically on M. Next, by using (2.10a) and taking into account that g is bundle–like we obtain

$$\widetilde{\nabla}_{Q'X}Q'X = \nabla^{\perp}_{Q'X}Q'X. \tag{4.21}$$

Finally, (4.21) and (2.9) imply

$$Q\widetilde{\nabla}_{QY}\widetilde{\nabla}_{Q'X}Q'X = Q\widetilde{\nabla}_{QY}\nabla^{\perp}_{Q'X}Q'X = -A_{\nabla^{\perp}_{Q'X}Q'X}QY = 0,$$

since $\mathcal{F}$ is totally geodesic. Hence (4.20) becomes

$$Q\widetilde{R}(QY, Q'X)Q'X = 0, \quad \text{on } M,$$

which contradicts the hypothesis of the theorem. Thus the proof is complete. ∎

In case (M, g) is a Riemannian manifold, from Theorem 4.12 we deduce the following.

Corollary 4.13. *Let $\mathcal{F}$ be a totally geodesic n-foliation of an $(n+p)$–dimensional, $p \geq 1$, Riemannian manifold such that g is bundle–like for $\mathcal{F}$. If all mixed sectional curvatures of M at a point x_0 are non-zero, then $n \leq p-1$.*

Corollary 4.14. *Let (M, g) be a positively or negatively curved semi–Riemannian manifold. Then we have the assertions:*

(i) *There exist no totally geodesic foliations with bundle–like metric on M such that $\mathcal{D}^{\perp}$ is integrable.*
(ii) *There exist no totally geodesic foliations with bundle–like metric and of codimension 1 on M.*
(iii) *If $\mathcal{F}$ is a totally geodesic foliation with bundle–like metric of codimension 2, then* $\dim M = 3$. *In particular, there exist no totally geodesic foliations with bundle–like metric of codimension 2 on spheres S^n with $n \geq 4$.*

Proof. Suppose there exists $\mathcal{F}$ satisfying conditions in (i). Then $\mathcal{D}^{\perp}$ defines a totally geodesic foliation $\mathcal{F}^{\perp}$ with bundle–like metric. Indeed, since $\mathcal{D}^{\perp}$ is integrable and $\mathcal{F}$ is bundle–like, we deduce that h' is both symmetric and skew-symmetric on $\mathcal{D}^{\perp}$. Hence $h' = 0$ on $\mathcal{D}^{\perp}$. Finally, because $h = 0$ on $\mathcal{D}$, it follows that $\mathcal{F}^{\perp}$ is with bundle–like metric. Thus, we may apply Theorem 4.12 for both $\mathcal{F}$ and $\mathcal{F}^{\perp}$, that is, $n \leq p-1$ and $p \leq n-1$, which lead to a contradiction. The assertion (ii) is a consequence of (i) since $\mathcal{D}^{\perp}$ is a line field in this case. Finally, if $\mathcal{F}$ is the one from assertion (iii) then by Theorem 4.12 we have $n \leq 1$, that is $n = 1$ and thus $\dim M = 3$. This completes the proof of the corollary. ∎

Other non-existence theorems on totally geodesic foliations can be found in Tondeur–Vanhecke [TV96].

By the next example we show that the above estimation for n is optimal.

Example 4.2. Let (M, g) be a real $(2n+1)$–dimensional Riemannian manifold endowed with a tensor field φ of type $(1,1)$, a vector field ξ and a 1–form η. Then we say that $(M, g, \varphi, \xi, \eta)$ is a **contact metric manifold** if these tensor fields satisfy

$$\text{(a) } \varphi^2 = -I + \eta \otimes \xi, \quad \text{(b) } \eta(X) = g(X, \xi), \quad \text{(c) } g(X, \varphi Y) = d\eta(X, Y), \tag{4.22}$$

for any $X, Y \in \Gamma(TM)$. By using (4.22) it is easy to check the following

$$\begin{aligned} &\text{(a) } \eta(\xi) = 1, \quad \text{(b) } g(X, \varphi Y) + g(Y, \varphi X) = 0, \\ &\text{(c) } g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{4.23}$$

for any $X, Y \in \Gamma(TM)$. Finally, we say that $(M, g, \varphi, \xi, \eta)$ is a **Sasakian manifold** if we have (cf. Blair [Bla76], p.73)

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad \forall\, X, Y \in \Gamma(TM), \tag{4.24}$$

where $\widetilde{\nabla}$ is the Levi–Civita connection on (M, g). In this case, two important identities are satisfied by the **characteristic vector field** ξ:

$$\text{(a) } \widetilde{\nabla}_\xi \xi = 0 \;\text{ and }\; \text{(b) } \widetilde{\nabla}_X \xi = -\varphi X, \quad \forall\, X \in \Gamma(TM). \tag{4.25}$$

By (4.25a) we see that ξ defines on (M, g) a totally geodesic 1-foliation $\mathcal{F}_\xi$. Moreover, by using (4.25b) and (4.23b) we deduce that

$$g(X, \widetilde{\nabla}_Y \xi) + g(Y, \widetilde{\nabla}_X \xi) = 0, \quad \forall\, X, Y \in \Gamma(TM), \tag{4.26}$$

that is ξ is a Killing vector field on (M, g). The **contact distribution** $\mathcal{D}^\perp$ on $(M, g, \varphi, \xi, \eta)$ is the complementary orthogonal distribution to $\mathcal{D} = \text{span}\{\xi\}$. Then from (4.26) it follows that , in particular, ξ is $\mathcal{D}^\perp$–Killing. Thus by assertion (iv) of Theorem 3.3 we deduce that g is bundle–like with respect to $\mathcal{F}_\xi$. Also, the curvature $\widetilde{R}$ of $\widetilde{\nabla}$ on a Sasakian manifold satisfies (cf. Blair [Bla76], p.74)

$$\widetilde{R}(\xi, X)X = \xi, \tag{4.27}$$

for any unit vector field $X \in \Gamma(\mathcal{D}^\perp)$. Thus in our notations $Q\widetilde{R}(\xi, X)X = \xi \neq 0$. There have been constructed Sasakian structures with interesting curvature properties on $\mathbb{R}^{2n+1}$ and on the unit sphere S^{2n+1} by Okumura [Oku62] and Tanno [Tan68], [Tan69]. Thus, summing up the above results we may state the following.

Theorem 4.15. *The foliation $\mathcal{F}_\xi$ determined by the characteristic vector field of a Sasakian manifold $(M, g, \varphi, \xi, \eta)$ is totally geodesic, with bundle–like metric and $Q\widetilde{R}(\xi, Q'X)Q'X \neq 0$.*

In particular, we deduce that $\mathcal{F}_\xi$ on both $\mathbb{R}^3$ and S^3 satisfies all the conditions from Theorem 4.14 with $n = p - 1 = 1$. Thus the estimation for n in Theorem 4.14 cannot be improved. Later in this book (see Section 5.2) we present some n–foliations on $(2n+1)$–dimensional contact manifolds. ∎

It is noteworthy that the important class of totally geodesic foliations with bundle–like metrics is characterized only by means of Vrănceanu connection as follows.

Theorem 4.16. *Let $\mathcal{F}$ be a non–degenerate n–foliation on an $(n+p)$–dimensional semi–Riemannian manifold (M, g). Then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is totally geodesic with bundle–like metric.*
(ii) *The Vrănceanu connection is a metric connection with respect to g.*

Proof. By condition (iii) of Theorem 4.2 and (3.1) we see that $\mathcal{F}$ is totally geodesic with bundle–like metric if and only if we have

$$(\nabla^*_X g)(QY, QZ) = 0 \quad \text{and} \quad (\nabla^*_X g)(Q'Y, Q'Z) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$, where ∇^* is the Vrănceanu connection induced by the Levi–Civita connection $\widetilde{\nabla}$ on (M, g). On the other hand, since ∇^* is an adapted linear connection on the almost product manifold $(M, \mathcal{D}, \mathcal{D}^\perp)$ (see Section 1.2), we have

$$\begin{aligned}(\nabla^*_X g)(QY, Q'Z) = \ & X(g(QY, Q'Z)) - g(\nabla^*_X QY, Q'Z) \\ & - g(QY, \nabla^*_X Q'Z) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM).\end{aligned}$$

Comparing the above equations satisfied by ∇^* and g with (1.5.9) we conclude that ∇^* is a metric connection with respect to g. Thus the proof is complete. ∎

An important question for foliations can be stated as follows. Given a foliation $\mathcal{F}$ on a manifold M, is there a Riemannian metric g on M such that $\mathcal{F}$ is totally geodesic? In the affirmative case $\mathcal{F}$ is called a **geodesible foliation** (cf. Johnson–Whitt [JW80]). When M is compact, Ghys [Ghy83] has classified the geodesible foliations of codimension 1. However in higher codimension the existence of geodesible foliations is still an open problem. From Theorem 4.16 it follows that the existence of geodesible foliations with bundle–like metric is equivalent to the existence of a Riemannian (semi–Riemannian) metric with respect to which the Vrănceanu connection is a metric connection.

Also, there were several investigations on totally geodesic foliations whose leaves are preserved by the flow of a Killing vector field. Important results on this problem have been obtained by Johnson and Whitt [JW80], Oshikiri [Osh83], [Osh86] and Curras–Bosch [CB88].

Now, we consider a foliation $\mathcal{F}$ on a Riemannian manifold (M, g) with bundle–like metric g. As we have seen at the end of the previous section, the geodesic symmetries of the transversal model Riemannian manifold need not be isometries. However, as the next theorem shows, this is true when both $\mathcal{F}$ and M satisfy some additional conditions.

Theorem 4.17. (Tondeur–Vanhecke [TV90]). *Let (M, g) be a Riemannian manifold of constant curvature, and $\mathcal{F}$ be a totally geodesic foliation on (M, g) with bundle–like metric g. Then $\mathcal{F}$ is transversally symmetric.*

According to the result stated in Theorem 4.12, we conclude that foliations from the above theorem must be of large codimension, when M has non–zero constant curvature.

At the end of Section 3.3 we have presented the transversally symmetric foliations in relation with locally symmetric Riemannian spaces. Here, we consider a class of totally geodesic foliations which is in relation with generalized symmetric Riemannian spaces. To introduce these concepts we start with a Riemannian manifold (M, g). An isometry of (M, g) with an isolated fixed point $x \in M$ is called a **symmetry** of (M, g) at x. A family $\{s_x : x \in M\}$ of symmetries of (M, g) is called an s–**structure** on (M, g). When each symmetry s_x is involutive we call $\{s_x : x \in M\}$ an **involutive** s–**structure** on (M, g). Then (M, g) is called a (**globally**) **symmetric Riemannian space** if it admits an involutive s–structure. The main results on the geometry of (locally or globally) symmetric Riemannian spaces can be found in the book of Helgason [Hel01].

Next, we say that the s–structure $\{s_x : x \in M\}$ is **regular** if it satisfies

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y),$$

for every two points $x, y \in M$. Then (M, g) is called a **generalized symmetric Riemannian space** (s–**manifold**) if it admits a regular s–structure (cf. Kowalski [Kow80], p. 8). A study of the geometry of generalized symmetric Riemannian spaces has been presented in the book of Kowalski [Kow80].

Now, let $\mathcal{F}$ be an n–foliation on an $(n+p)$–dimensional Riemannian manifold. Denote by G the group of all leaf–preserving isometries of M. Thus $f \in G$ if and only if f is an isometry of M, and for each $x \in M$, $f(L_x) = L_y$, where $y = f(x)$ and L_x, L_y are the leaves through x and y respectively. If $\theta : M \longrightarrow \widetilde{M}$ denotes the continuous projection to the leaf space $\widetilde{M} = M/\mathcal{F}$, then an isometry f of M is an element of G if and only if there is a homeomorphism $\widetilde{f} : \widetilde{M} \longrightarrow \widetilde{M}$, necessarily unique, such that $\theta \circ f = \widetilde{f} \circ \theta$. A leaf L is said to be a **fixed leaf** of f if $f(x) = x$ for every $x \in L$. That is, L is a fixed leaf of f, if and only if L is point–wise fixed by f. Finally, a fixed leaf L is said to be an **isolated fixed leaf** if L is an isolated fixed point of $\widetilde{M}$.

With the concept of generalized symmetric Riemannian space in mind, we say that $\mathcal{F}$ is a **symmetric foliation** of M if, for each leaf L of $\mathcal{F}$, there is a leaf–preserving isometry f_L of M for which L is an isolated fixed leaf. Thus the concept of symmetric foliation, introduced by Farran and Robertson [FR96], reduces to that of generalized symmetric Riemannian space when the foliation of the manifold is the trivial foliation of M by point leaves. A family of examples of symmetric foliations is provided by the Hopf fiberings of the $(2n+1)$–dimensional sphere S^{2n+1} over the complex projective space $P_n(\mathbb{C})$ by great circles, and of the sphere S^{4n+3} over the quaternionic projective space $P_n(\mathbb{H})$ by great 3–spheres. We present other examples after the next theorems which have been proved by Farran and Robertson [FR96].

Theorem 4.18. *Every symmetric foliation $\mathcal{F}$ of a Riemannian manifold (M, g) is totally geodesic. Moreover, every leaf of $\mathcal{F}$ is a closed subset of M.*

Theorem 4.19. *Let $\mathcal{F}$ be a symmetric foliation of a Riemannian manifold (M, g). Then the leaf space $\widetilde{M} = M/\mathcal{F}$ has a structure of a smooth, possibly non–Hausdorff manifold for which $\theta : M \to \widetilde{M}$ is a submersion.*

Theorem 4.20. *Let $\mathcal{F}$ be a symmetric foliation with compact leaves of a Riemannian manifold (M, g). Then we have the following:*

(i) *$\theta : M \to \widetilde{M}$ is a fibering, provided M is complete.*
(ii) *The group G of leaf–preserving isometries of M acts transitively on $\widetilde{M}$.*
(iii) *If $\widetilde{g}$ is a G-invariant Riemannian metric on $\widetilde{M}$, then $(\widetilde{M}, \widetilde{g})$ is a generalized symmetric Riemannian manifold.*

The two notions of transversally symmetric foliation (see Section 3.3) and symmetric foliation on a Riemannian manifold have been introduced independently in Tondeur–Vanhecke [TV90] and Farran–Robertson [FR96] respectively. Now, we would like to discuss the relationship between these two classes of foliations. The following remarks will explain this relationship.

Remark 4.3. Transversally symmetric foliations need not be symmetric foliations. The next example supports our assertion. ∎

Example 4.4. let $\mathcal{F}$ be the foliation of $M = \mathbb{R}^2 \backslash \{0\}$ by circles with center at the origin, and g be the Euclidean metric on M. Then it is easy to see that g is a bundle–like metric for $\mathcal{F}$. Also, since $\mathcal{F}$ is of codimension one, by Corollary 3.14 we conclude that $\mathcal{F}$ is transversally symmetric. However $\mathcal{F}$ is not a symmetric foliation because its leaves are not totally geodesic submanifolds of M. ∎

Also, the foliation in Example 4.4.1 supports the above assertion. Indeed, that foliation is with bundle–like metric (being parallel) and it is transversally symmetric (being of codimension one). However, it is not a symmetric foliation because its leaves are not closed subsets of the torus.

Remark 4.5. Symmetric foliations need not be transversally symmetric foliations. We give two examples to support this assertion. ■

Example 4.6. Let $\mathcal{F}$ be the foliation of $M = \mathrm{I\!R}^2 \setminus \{0\}$ by straight rays emanating from the origin, and g be the Euclidean metric on M. Every leaf (ray) L determines a unique straight line S_L through the origin. We take f_L to be the reflection with respect to S_L. Then f_L is a leaf–preserving isometry for which L is an isolated fixed leaf. Thus $\mathcal{F}$ is a symmetric foliation. However, $\mathcal{F}$ is not a transversally symmetric foliation because g is not bundle–like for $\mathcal{F}$.■

Example 4.7. Let (P, h) be a generalized symmetric Riemannian space which is not a locally symmetric Riemannian space, and (P', h') be any Riemannian manifold. Take (M, g) to be the Riemannian product $(P, h)\times(P', h')$ and $\mathcal{F}$ the foliation of M by copies of (P', h'). Then the leaf space of $\mathcal{F}$ can be identified with (P, h) and therefore $\mathcal{F}$ is a symmetric foliation. However $\mathcal{F}$ is not transversally symmetric because the transversal model of $\mathcal{F}$ is not a locally symmetric Riemannian space. ■

3.4.2 Totally Umbilical Foliations on Semi–Riemannian Manifolds

Let $\mathcal{F}$ be a non–degenerate n–foliation on an $(n + p)$–dimensional semi–Riemannian manifold (M, g). Consider the second fundamental form h of $\mathcal{F}$ given by (2.5) and choose an orthonormal frame field $\{E_1, ..., E_n\}$ of signature $\{\varepsilon_1, ..., \varepsilon_n\}$ in $\Gamma(\mathcal{D})$, where $\mathcal{D}$ is the structural distribution of $\mathcal{F}$. Then we define the **mean curvature vector field** H of $\mathcal{F}$ by the formula

$$H = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i h(E_i, E_i). \tag{4.28}$$

It is easy to check that H does not depend on the orthonormal basis $\{E_i\}$, so it is a global section of the transversal distribution $\mathcal{D}^{\perp}$. If $\{E_\alpha\}$, $\alpha \in \{n + 1, ..., n + p\}$ is an orthonormal basis with signature $\{\varepsilon_\alpha\}$ in $\Gamma(\mathcal{D}^{\perp})$, we denote by A_α the shape operators of $\mathcal{F}$ with respect to E_α (see (2.9)). Then by using (3.45) and (2.12c) we express H as follows

$$H = \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \sum_{i=1}^{n} \varepsilon_\alpha \varepsilon_i g(A_\alpha E_i, E_i) E_\alpha. \tag{4.29}$$

We note that nH is denoted in Kamber–Tondeur [KT82] by τ and it is called the **tension field** of $\mathcal{F}$. The **mean curvature form** of the foliation $\mathcal{F}$ on (M, g) is a 1–form k on M defined by

$$k(X) = g(X, H), \quad \forall\, X \in \Gamma(TM). \tag{4.30}$$

Thus we have

$$\text{(a) } k(QX) = 0 \text{ and (b) } k(Q'X) = g(Q'X, H), \quad \forall X \in \Gamma(TM). \tag{4.31}$$

By using (4.28) and (4.29) in (4.31b) we deduce that

$$\begin{aligned} k(Q'X) &= \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i g(h(E_i, E_i), Q'X) \\ &= \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \sum_{i=1}^{n} \varepsilon_\alpha \varepsilon_i g(A_\alpha E_i, E_i) g(E_\alpha, Q'X). \end{aligned} \tag{4.32}$$

Now, let $\left\{ \frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^\alpha} \right\}$ be a semi–holonomic frame field on the foliated semi–Riemannian manifold $(M, g, \mathcal{F})$ (see (1.21)). Then we put

$$\text{(a) } H = H^\alpha \frac{\delta}{\delta x^\alpha} \text{ and (b) } k_\alpha = k\left(\frac{\delta}{\delta x^\alpha}\right) \cdot \tag{4.33}$$

Thus H (resp. k) is a transversal vector field (resp. transversal 1–form), and on the domain of a foliated chart on M we have

$$k_\alpha = g_{\alpha\beta} H^\beta. \tag{4.34}$$

Next, we suppose that at any point of M, H is not a light–like vector. In this case we consider the unit vector field N in the direction of H, that is, we have

$$N = \frac{1}{\|H\|} H. \tag{4.35}$$

Then we define the **mean curvature function** τ of the foliation $\mathcal{F}$ with respect to N by

$$\tau = nk(N) = ng(N, H). \tag{4.36}$$

By using (4.28), (2.5) and (4.35) in (4.36) we obtain

$$\begin{aligned} &\text{(a) } \tau = \sum_{i=1}^{n} \varepsilon_i g(N, h(E_i, E_i)) = \sum_{i=1}^{n} \varepsilon_i g(N, \widetilde{\nabla}_{E_i} E_i), \\ &\text{(b) } \tau = n\varepsilon_N \|H\|, \end{aligned} \tag{4.37}$$

where $\varepsilon_N = \pm 1$ is the signature of N and $\widetilde{\nabla}$ is the Levi–Civita connection on (M, g). In the Riemannian case (4.37a) becomes (see Oshikiri [Osh90])

$$\tau = \sum_{i=1}^{n} g(N, \widetilde{\nabla}_{E_i} E_i). \tag{4.38}$$

For a foliation of codimension one the choice of N as in (4.35) gives an orientation for the transversal distribution. In this case there were found conditions for a smooth function on M in order to be represented as a mean curvature function with respect to a metric on M (cf. Walczak [Wa84], Oshikiri [Osh90], [Osh91]).

Now, we come back to the general case and give the following definition. We say that a non–degenerate foliation $\mathcal{F}$ on a semi–Riemannian manifold (M, g) is **totally umbilical** if its second fundamental form h given by (2.5) satisfies

$$h(QX, QY) = g(QX, QY)H, \quad \forall\, X, Y \in \Gamma(TM), \tag{4.39}$$

where H is the mean curvature vector field of $\mathcal{F}$. Clearly $\mathcal{F}$ is totally umbilical if and only if its leaves are totally umbilical (see O'Neill [O83], p. 106). In particular, from (4.39) we obtain

$$h(QX, QX) = g(QX, QX)H, \quad \forall\, X \in \Gamma(TM),$$

which says that the leaves of $\mathcal{F}$ bend toward H in space–like directions and away from H in time–like directions.

The condition (4.39) can also be expressed by using the shape operator of the foliation. Indeed, by using (2.12c) and (4.39) we obtain

$$g(A_{Q'Z}QX, QY) = g(h(QX, QY), Q'Z) = g(g(H, Q'Z)QX, QY).$$

Thus $\mathcal{F}$ is totally umbilical if and only if its shape operators satisfy

$$A_{Q'Z}QX = k(Q'Z)QX, \quad \forall\, X, Z \in \Gamma(TM). \tag{4.40}$$

Now, we put $A_\alpha = A_{\frac{\delta}{\delta x^\alpha}}$, $\alpha \in \{n+1, ..., n+p\}$ and by using (4.40) obtain the following.

Theorem 4.21. *A non–degenerate foliation $\mathcal{F}$ on a semi–Riemannian manifold (M, g) is totally umbilical if and only if its shape operators A_α satisfy*

$$A_\alpha = k_\alpha I, \;\; \alpha \in \{n+1, ..., n+p\}, \tag{4.41}$$

where I is the identity on $\Gamma(\mathcal{D})$ and k_α are the local components of the mean curvature form given by (4.34).

Proposition 4.22. *Any non–degenerate* 1*–foliation on a semi–Riemannian manifold (M, g) is totally umbilical.*

Proof. Let E_1 be a unit vector field spanning $\mathcal{D}$ in a certain neighbourhood. Then by (4.28) we have

$$H = \varepsilon_1 h(E_1, E_1).$$

Thus for any $X \in \Gamma(\mathcal{D})$ we have $X = fE_1$ and

$$h(X, X) = f^2 h(E_1, E_1) = f^2 \varepsilon_1 H = f^2 g(E_1, E_1)H = g(X, X)H,$$

which proves our assertion. ■

Example 4.8. Let $\mathbb{R}^{n+1}_q$, $n \geq 2$, be the $(n+1)$–dimensional semi–Euclidean space of index $0 \leq q \leq n$. Then the **pseudo–sphere** of radius $r > 0$ in $\mathbb{R}^{n+1}_q$ is the hyperquadric defined by

$$S^n_q(r) = \{x \in \mathbb{R}^{n+1}_q : g(x, x) = r^2\},$$

where g is given by (1.4.9). Similarly, the **pseudo–hyperbolic space** of radius $r > 0$ in $\mathbb{R}^{n+1}_{q+1}$ is the hyperquadric

$$H^n_q(r) = \{x \in \mathbb{R}^{n+1}_{q+1} : g(x, x) = -r^2\}.$$

It is known (see O'Neill [O83], p.111) that both $S^n_q(r)$ and $H^n_q(r)$ are totally umbilical hypersurfaces of $\mathbb{R}^{n+1}_q$ and $\mathbb{R}^{n+1}_{q+1}$ respectively. Therefore the set of all pseudo–spheres in $\mathbb{R}^{n+1}_q$ (resp. pseudo–hyperbolic spaces in $\mathbb{R}^{n+1}_{q+1}$) defines a totally umbilical foliation on $M = \mathbb{R}^{n+1}_q \backslash \{0\}$ (resp. $M = \mathbb{R}^{n+1}_{q+1} \backslash \{0\}$). In particular, the set of all spheres centered at the origin defines a totally umbilical foliation on $\mathbb{R}^{n+1} \backslash \{0\}$. ■

Let M be a non–degenerate real hypersurface of an indefinite almost Hermitian manifold $(\widetilde{M}, J, g)$. If $TM^{\perp}$ is the normal bundle of M, then $J(TM^{\perp})$ defines a line field on M. Thus, by Proposition 4.22, M carries a totally umbilical 1–foliation. In particular, any non–degenerate hypersurface of $\mathbb{R}^{2n}_q$ is endowed with a totally umbilical 1–foliation.

To state some results on the geometry of totally umbilical foliations we give the following definitions. We say that the foliation $\mathcal{F}$ is **homogeneous** if it is an orbit foliation of a group of isometries. When the transversal distribution $\mathcal{D}^{\perp}$ to a foliation $\mathcal{F}$ defines a totally geodesic foliation we say that $\mathcal{F}$ is **flat**. It was proved by Gromoll and Grove [GG85] that line fields with bundle–like metrics (**Riemannian flows**) are always flat or homogeneous in any space of constant curvature. This result was generalized by Walschap [Was97] to totally umbilical foliations as follows.

Theorem 4.23. (Walschap [Was97]). *Let $\mathcal{F}$ be a totally umbilical n–foliation, $n > 1$, on a complete simply connected Riemannian manifold $(M(c), g)$ of constant curvature c, such that g is bundle–like for $\mathcal{F}$. Then $\mathcal{F}$ is flat if $c \leq 0$ and homogeneous (actually totally geodesic) if $c \geq 0$.*

The proof of this theorem is based on the Riccati type equation (4.16) and we omit it here.

Next, we want to get more information about the geometry of $\mathcal{F}$ on constant curvature manifolds. First we recall that the curvature tensor field $\widetilde{R}$ of $(M(c), g)$ is given by (cf. Chen [C73], p. 47)

$$\widetilde{R}(X, Y)Z = c(g(Z, Y)X - g(Z, X)Y), \quad \forall X, Y, Z \in \Gamma(TM). \tag{4.42}$$

The case $c \geq 0$ is the most simple. Indeed, since $\mathcal{F}$ is a totally geodesic foliation, by using (4.1) and (4.42) in (1.6.3) we deduce that all leaves of $\mathcal{F}$ are also Riemannian manifolds of constant curvature c. To study the case $c \leq 0$ we first consider the curvature tensor field R of the induced connection ∇ on the structural distribution $\mathcal{D}$ (see (2.3a)). Then we say that $\mathcal{F}$ is a foliation of **scalar curvature** K if R can be expressed as follows on $\Gamma(\mathcal{D})$

$$R(QX, QY)QZ = K(g(QZ, QY)QX - g(QZ, QX)QY), \tag{4.43}$$

for any $X, Y, Z \in \Gamma(TM)$. As the restriction of R to each leaf of $\mathcal{F}$ is just the curvature tensor field of that leaf, by Schur Theorem for Riemannian manifolds (cf. Kobayashi–Nomizu [KN63], p.202) we conclude that the function K from (4.43) must be basic, that is, K depends on $(x^{n+1}, ..., x^{n+p})$ alone provided $n > 2$. Now, by using (4.39), (4.42) and (4.30) in (1.6.3) we obtain

$$R(QX, QY)QZ = (c + k(H))(g(QZ, QY)QX - g(QZ, QX)QY), \tag{4.44}$$

where $k(H)$ depends on (x^α) alone, if $n > 2$. Next, from (1.6.13) we deduce that the torsion tensor field of the Schouten–Van Kampen connection satisfies

$$T^\circ(QX, QY) = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{4.45}$$

Then we take $X = QX$ and $Y = QY = QZ$ in (1.6.4) and by using (4.42), (4.39), (4.45) and taking into account that ∇ is a metric connection (cf. (i) of Lemma 1.5.5) we obtain

$$\nabla^\perp_{QX} H = 0, \quad \forall\, X \in \Gamma(TM), \tag{4.46}$$

which implies that $k(H)$ is basic in any dimension. When $k(H)$ is a constant on M, that is all leaves of $\mathcal{F}$ have the same constant curvature, we say that the foliation is of **constant curvature**. Now we state the following.

Theorem 4.24. *Let $\mathcal{F}$ be a totally umbilical n–foliation, $n > 1$, on a complete simply connected Riemannian manifold $(M(c), g)$ with bundle–like metric and $c \leq 0$. Then we have the assertions:*

(i) *$\mathcal{F}$ is of scalar curvature $c + k(H)$, where $k(H)$ is a basic function.*
(ii) *$\mathcal{F}$ is of constant curvature if and only if all leaves of $\mathcal{F}$ are flat, that is, they are of zero sectional curvature.*

Proof. Clearly, the first assertion follows from the arguments stated before the theorem.

Next, by using (4.42) and (2.12b) in (1.6.3) and taking into account that $\mathcal{F}$ is flat we obtain

$$\begin{aligned} &g(R(Q'X, QY)QZ, QU) = g(h(QY, QZ), h(Q'X, QU)) \\ &-g(h(Q'X, QZ), h(QY, QU)) = g(h'(Q'X, h(QY, QU)), QZ) \\ &-g(h'(Q'X, h(QY, QZ)), QU) = 0, \end{aligned}$$

since h' vanishes on $\mathcal{D}^{\perp}$. Thus $R(Q'X, QY)QZ = 0$, which in local coordinates means (see (2.3.30))

$$R_i{}^h{}_{\gamma j} = 0. \tag{4.47}$$

Then by direct calculations using (2.17), (4.39), (4.30) and (4.33b) we deduce that

$$\text{(a) } h_\alpha{}^\beta{}_i = 0 \;\text{ and }\; \text{(b) } h'_i{}^k{}_\alpha = -k_\alpha \delta_i^k. \tag{4.48}$$

Now we use the Bianchi identity (2.4.31) for the Schouten–Van Kampen connection ∇° on $(M(c), g)$. By using Proposition 2.9 and (4.48) we deduce that the adapted torsion tensor fields from (2.4.31) are given by

$$\begin{gathered}\text{(a) } T_j{}^r{}_k = T^\circ{}_j{}^r{}_k = 0, \quad \text{(b) } T_\gamma{}^r{}_j = -h'_j{}^r{}_\gamma = k_\gamma \delta_j^r, \\ \text{(c) } C'_\gamma{}^\varepsilon{}_k = T^\circ{}_\gamma{}^\varepsilon{}_k = h_\gamma{}^\varepsilon{}_k = 0.\end{gathered} \tag{4.49}$$

Thus by (4.47) and (4.49) we see that (2.4.31) becomes

$$R_i{}^h{}_{jk|^\circ\gamma} - 2k_\gamma R_i{}^h{}_{jk} = 0, \tag{4.50}$$

where $|^\circ$ represents transversal covariant derivative with respect to ∇° (see Section 3.2) and $R_i{}^h{}_{jk}$ are the local components of R from (4.44) with respect to the semi–holonomic frame field $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\delta}{\delta x^\alpha}\right\}$, i.e., we have

$$R_i{}^h{}_{jk} = (c + k(H))(g_{ij}\delta_k^h - g_{ik}\delta_j^h). \tag{4.51}$$

Taking the transversal covariant derivative in (4.51) and by using (2.19b) and (2.3.17) we obtain

$$R_i{}^h{}_{jk|^\circ\gamma} = \frac{\partial(k(H))}{\partial x^\gamma}(g_{ij}\delta_k^h - g_{ik}\delta_j^h), \tag{4.52}$$

since $k(H)$ is a basic function on $M(c)$. Comparing (4.50) with (4.52) and using (4.51) we deduce that

$$\left(\frac{\partial(k(H))}{\partial x^\gamma} - 2k_\gamma(c + k(H))\right)(n-1)\delta_k^h = 0,$$

which implies

$$\frac{\partial(k(H))}{\partial x^\gamma} = 2k_\gamma(c + k(H)), \tag{4.53}$$

since $n > 1$. Now, suppose that $\mathcal{F}$ is of constant curvature. Then from (4.44) it follows that $c + k(H)$ must be a constant on M. As $k(H)$ is basic, from (4.53) we obtain

$$\text{(a) } c + k(H) = 0 \;\text{ or }\; \text{(b) } k_\alpha = 0, \quad \text{for all } \alpha \in \{n+1, ..., n+p\}.$$

But (b) can not occur because $k_\alpha = 0$ implies $H = 0$ and thus $\mathcal{F}$ is totally geodesic with bundle–like metric on $M(c)$. As by Theorem 4.23 $\mathcal{F}$ is flat, we apply assertion (i) of Corollary 4.14 and justify our assertion. Thus only (a) can occur, and this proves the assertion (ii) of our theorem. ■

In case of positively curved manifolds a similar result to the one stated in Corollary 4.13 for totally geodesic foliations has been obtained.

Theorem 4.25. (Walschap [Was97]). *Let $\mathcal{F}$ be a totally umbilical n–foliation with bundle–like metric on an $(n+p)$–dimensional Riemannian manifold (M, g) of positive curvature. Then $n \leq p - 1$.*

Finally, we mention here that a foliation $\mathcal{F}$ on a manifold M is said to be **umbilicalizable** if there exists a Riemannian (semi–Riemannian) metric g on M for which $\mathcal{F}$ is totally umbilical. As it is well known, any totally umbilical and minimal non–degenerate submanifold of a semi–Riemannian manifold is totally geodesic. However, such an assertion on umbilicalizable and geodesible foliations is not obvious. Results on this matter can be found in Carrière [Car81] and Cairns [Cai90]. Also, some decomposition theorems for Riemannian manifolds endowed with two complementary orthogonal totally umbilical foliations have been obtained by Koike [K90].

3.4.3 Minimal Foliations on Riemannian Manifolds

Let (M, g) be an $(n+p)$–dimensional semi–Riemannian manifold and $\mathcal{F}$ be a non–degenerate n–foliation on M with $\mathcal{D}$ and $\mathcal{D}^\perp$ as structural and transversal distributions respectively. Consider the Levi–Civita connection $\widetilde{\nabla}$ on (M, g) and the intrinsic connection $D^\perp$ on $\mathcal{D}^\perp$ defined by (1.10b). Now, we define a $\mathcal{D}^\perp$**–valued differential** r**–form** on M, as an $F(M)$–multilinear mapping $\omega : \Gamma(TM)^r \longrightarrow \Gamma(\mathcal{D}^\perp)$ such that

$$\omega(X_{\sigma(1)}, ..., X_{\sigma(r)}) = \varepsilon(\sigma)\omega(X_1, ..., X_r),$$

for any permutation σ of $\{1, 2, ..., r\}$, where $\varepsilon(\sigma) = \pm 1$ is the signature of σ. Then we define the **intrinsic covariant derivative** of ω with respect to $X \in \Gamma(TM)$ as the r–form $D^\perp_X \omega$ given by

$$(D^\perp_X \omega)(Y_1, ..., Y_r) = D^\perp_X(\omega(Y_1, ..., Y_r)) - \sum_{i=1}^{r} \omega(Y_1, ..., \widetilde{\nabla}_X Y_i, ..., Y_r), \quad (4.54)$$

for any $Y_i \in \Gamma(TM)$, $i \in \{1, ..., r\}$. Next, denote by $\mathcal{A}^r(M, \mathcal{D}^\perp)$ the $F(M)$–module of all $\mathcal{D}^\perp$–valued differential r–forms on M. Then we define the $D^\perp$**–exterior derivative** as the differential operator

$$d : \mathcal{A}^r(M, \mathcal{D}^\perp) \longrightarrow \mathcal{A}^{r+1}(M, \mathcal{D}^\perp),$$

given by

$$d\omega(Y_1, ..., Y_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} (D^{\perp}_{Y_i}\omega)(Y_1, ..., \widehat{Y}_i, ..., Y_{r+1}), \tag{4.55}$$

where $\widehat{Y}_i$ means that Y_i is omitted. As in Section 3.1 denote by R' the curvature tensor field of $D^{\perp}$ and keep the same symbol for the $L(\mathcal{D}^{\perp}, \mathcal{D}^{\perp})$–valued 2–form

$$(X, Y) \longrightarrow R'(X, Y), \quad \forall\, X, Y \in \Gamma(TM).$$

Then it can be proved that

$$d^2\omega = R' \wedge \omega,$$

where $\wedge$ is the usual exterior product of vector bundles valued forms. According to (1.33) we have $R'(QX, QY) = 0$ for any $X, Y \in \Gamma(TM)$, and therefore $d^2\omega = 0$ for any ω restricted to $\Gamma(\mathcal{D})^r$. Thus a De Rham cohomology of $\mathcal{D}^{\perp}$–valued forms along the leaves can be developed. This was done in more general setting by using the quotient bundle $TM/\mathcal{D}$ by several authors (cf. Vaisman [Vai71], Kamber–Tondeur [KT71]).

Next, we consider the $D^{\perp}$–**divergence operator**

$$d^* : \mathcal{A}^r(M, \mathcal{D}^{\perp}) \longrightarrow \mathcal{A}^{r-1}(M, \mathcal{D}^{\perp}),$$

given by

$$d^*\omega(Y_1, ..., Y_{r-1}) = -\sum_{A=1}^{n+p} \varepsilon_A (D^{\perp}_{E_A}\omega)(E_A, Y_1, ..., Y_{r-1}), \tag{4.56}$$

where $\{E_A\}$, $A \in \{1, ..., n+p\}$ is an orthonormal field of frames on (M, g) of signature $\{\varepsilon_A\}$, adapted to the decomposition (1.1). We note that $\mathcal{A}^0(M, \mathcal{D}^{\perp})$ is identified with $\Gamma(\mathcal{D}^{\perp})$.

More about the above three operators $D^{\perp}, d, d^*$ on Riemannian manifolds can be found in Tondeur [Ton97], where $D^{\perp} = \nabla$, $d = d_{\nabla}$ and $d^* = \delta_{\nabla}$. Also, in the Riemannian case the name divergence operator was given to d^* by Sanini [San82].

Our purpose is to present the basic properties of these operators on semi–Riemannian manifolds. To this end we consider the projection morphism $Q' : \Gamma(TM) \longrightarrow \Gamma(\mathcal{D}^{\perp})$ as a $\mathcal{D}^{\perp}$–valued 1–form. Then we apply $D^{\perp}$, d and d^* to Q' and by using (4.54), (4.55) and (4.56) we obtain

$$(D^{\perp}_X Q')(Y) = D^{\perp}_X Q'Y - Q'(\widetilde{\nabla}_X Y), \tag{4.57}$$

$$dQ'(X, Y) = (D^{\perp}_X Q')(Y) - (D^{\perp}_Y Q')(X), \tag{4.58}$$

and

$$d^*Q' = -\sum_{A=1}^{n+p} \varepsilon_A (D^{\perp}_{E_A} Q')(E_A), \tag{4.59}$$

respectively. According to the name we gave to d^*, we also call d^*Q' the **divergence** of Q'. Now, we prove the following.

Lemma 4.26. *Let $\mathcal{F}$ be a non–degenerate n–foliation on an $(n+p)$–dimensional semi–Riemannian manifold (M, g). Then we have*

$$\begin{aligned} &\text{(a) } (D^{\perp}_X Q')(QY) = -h(X, QY), \\ &\text{(b) } (D^{\perp}_X Q')(Q'Y) = -h(Q'Y, QX), \\ &\text{(c) } dQ' = 0, \quad \text{(d) } d^*Q' = nH, \end{aligned} \tag{4.60}$$

where H is the mean curvature vector field of $\mathcal{F}$ and h is the $F(M)$–bilinear form given by (2.4a).

Proof. By using (4.57) and (2.4a) we obtain (4.60a). Then we use (4.57), (2.3b) and (2.7d) and deduce (4.60b). Next, taking into account (4.58), (1.5.8), (1.4) and (1.9) we infer that

$$\begin{aligned} dQ'(X, Y) &= D^{\perp}_X Q'Y - D^{\perp}_Y Q'X - Q'[X, Y] \\ &= \{D^{\perp}_X Q'Y - D^{\perp}_{Q'Y} Q'X - Q'[X, Q'Y]\} \\ &- Q'[QY, Q'X] - Q'[X, QY] \\ &= -Q'\{[QY, Q'X] + [Q'X, QY]\} - Q'[QX, QY] = 0, \end{aligned}$$

which proves (4.60c). Finally, we use (4.60a), (4.60b) and (4.28) into (4.59) and obtain

$$\begin{aligned} d^*Q' &= -\sum_{i=1}^{n} \varepsilon_i \left(D^{\perp}_{E_i} Q'\right) E_i - \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha \left(D^{\perp}_{E_\alpha} Q'\right) E_\alpha \\ &= \sum_{i=1}^{n} \varepsilon_i h(E_i, E_i) + \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha h(Q'E_\alpha, QE_\alpha) \\ &= \sum_{i=1}^{n} \varepsilon_i h(E_i, E_i) = nH, \end{aligned}$$

that is, (4.60d) is proved. ■

For foliations on Riemannian manifolds the proofs of (4.60c) and (4.60d) were given by Kamber and Tondeur [KT82] (cf. Propositions 2.2 and 3.2).

From (4.60c) we see that the exterior derivative of Q' vanishes identically on M, while its intrinsic covariant derivative and divergence, in general, do not. When this happens the foliation has some special geometric properties as we see in the next two theorems.

Theorem 4.27. *Let $\mathcal{F}$ be a foliation as in Lemma* 4.26*. Then the intrinsic covariant derivative of Q' vanishes identically on M if and only if (M, g) is a locally semi–Riemannian product with respect to the decomposition* (1.1).

Proof. From (4.60a) and (4.60b) we deduce that $D^{\perp}_X Q' = 0$ for any $X \in \Gamma(TM)$ if and only if

$$\text{(a) } h(QX, QY) = 0 \quad \text{and} \quad \text{(b) } h(Q'X, QY) = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{4.61}$$

Next, by using (2.12b) we see that (4.61b) is equivalent to

$$h'(Q'X, Q'Z) = 0, \forall\, X, Z \in \Gamma(TM).$$

So $D^{\perp}_X Q' = 0$ for any $X \in \Gamma(TM)$ if and only if the second fundamental forms h and h' of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ vanish identically on M, that is, M is a locally semi–Riemannian product (see Section 1.5). ■

Theorem 4.28. *Let $\mathcal{F}$ be a foliation as in Lemma 4.26. Then the following assertions are equivalent:*

(i) *The divergence of Q' vanishes identically on M.*
(ii) *The mean curvature vector H of $\mathcal{F}$ vanishes identically on M.*
(iii) *The mean curvature form k of $\mathcal{F}$ vanishes identically on M.*

Proof. The equivalence of (i) and (ii) follows from (4.60d). Also, (4.30) implies the equivalence of (ii) and (iii). ■

From now on, we restrict our study to foliations on Riemannian manifolds. In this case, the equivalence of (i) and (ii) in Theorem 4.28 has been proved by Kamber and Tondeur [KT82]. If one of the assertions in Theorem 4.28 is satisfied (and therefore all) we say that $\mathcal{F}$ is a **minimal foliation** or a **harmonic foliation**. By assertion (ii) we see that $\mathcal{F}$ is minimal if and only if all leaves of $\mathcal{F}$ are minimal submanifolds of (M, g). This gives us a reason to call $\mathcal{F}$ a minimal foliation. Also by (i) we see that if $\mathcal{F}$ is harmonic then the Laplacian of Q' given by $\Delta Q' = dd^*Q' + d^*dQ'$ vanishes via (4.60c). Thus Q' is a harmonic $\mathcal{D}^{\perp}$–valued 1–form, which justifies the name harmonic for $\mathcal{F}$. When (M, g) is compact and oriented, and g is bundle–like for $\mathcal{F}$, it was proved by Kamber and Tondeur [KT82] that $\Delta Q' = 0$ implies $d^*Q' = 0$.

Proposition 4.29. *Let $\mathcal{F}$ be a foliation on a Riemannian manifold (M, g) such that:*

(i) *The mean curvature vector is parallel with respect to the intrinsic connection $D^{\perp}$ on $\mathcal{D}^{\perp}$, i.e., we have*

$$D^{\perp}_X H = 0, \quad \forall\, X \in \Gamma(TM). \tag{4.62}$$

(ii) *The transversal Ricci tensor of $\mathcal{F}$ is non–degenerate on M.*

Then $\mathcal{F}$ is a minimal foliation.

Proof. Locally (4.62) is equivalent to

$$\text{(a) } H^{\alpha}{}_{\|k} = 0, \quad \text{(b) } H^{\alpha}{}_{|\beta} = 0. \tag{4.63}$$

Then by using (4.63) and (1.38b) in (1.50) we deduce that

$$H^{\varepsilon} R'_{\varepsilon\beta} = 0,$$

which implies $H^{\varepsilon} = 0$ for any $\varepsilon \in \{n+1, ..., n+p\}$, since $[R'_{\alpha\beta}]$ is non–degenerate. Hence $\mathcal{F}$ is minimal. ∎

Remark 4.9. The condition (ii) is satisfied by any foliation that is transversal Einstein with non-zero transversal scalar curvature. Also, we have the same conclusion if we replace (ii) by the topological condition (see Kamber–Tondeur [KT82])

(ii′) M is a compact and oriented manifold. ∎

As in the case of the other two classes of foliations studied in the previous subsections, there were several studies on the existence of a Riemannian metric g on M with respect to which the foliation $\mathcal{F}$ is minimal. In the affirmative case the foliation is called **geometrical taut**. It was first Sullivan [Sul79] who found a necessary and sufficient condition for $\mathcal{F}$ to be taut. Then Rummler [Rum79] and Haefliger [Hae80] obtained geometrical and topological characterizations of taut foliations.

Finally, we note that the three problems on the existence of a Riemannian metric g with respect to which $\mathcal{F}$ falls into one of the categories: totally geodesic, totally umbilical or minimal become much more difficult in the semi–Riemannian case. This is because the existence of such a metric requires some strong topological conditions. For instance, a Lorentz metric exists on M if and only if either M is noncompact, or M is compact and has Euler number $\chi(M) = 0$ (cf. O'Neill [O83], p. 149). In general, there is a close relationship between the existence of a semi–Riemannian metric of index q on a manifold M and the existence of a q–distribution on M. More precisely, a smooth compact manifold admits a semi–Riemannian metric of index q if and only if it admits a q–distribution (see Steenrod [Stee51], p. 207). This explains the above results on the Euler number of a compact manifold endowed with a Lorentz metric.

3.5 Degenerate Foliations of Codimension One

In the present section we initiate a study of the geometry of a degenerate foliation of codimension one on a semi–Riemannian manifold. We introduce the concept of screen distribution on a manifold endowed with a degenerate foliation and construct the null transversal bundle to the foliation. Though this bundle depends on the screen distribution, the second fundamental form of a degenerate foliation is the same for all screen distributions.

Let (M, g) be an $(n+1)$–dimensional proper semi–Riemannian manifold and $\mathcal{F}$ be an n–foliation with structural and transversal distribution $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. Suppose that the null distribution $\mathcal{N} = \mathcal{D} \cap \mathcal{D}^{\perp}$ is of maximum rank 1, that is, all fibers of $\mathcal{N}$ are 1–dimensional. Then we say that $\mathcal{F}$ is a **degenerate foliation** on (M, g). In this case $\mathcal{N} = \mathcal{D}^{\perp}$ and thus $\mathcal{F}$ is degenerate if and only if $\mathcal{D}^{\perp}$ is a subbundle of $\mathcal{D}$. On the other hand, if $\mathcal{F}$ is degenerate then the induced tensor field by g on $\mathcal{D}$ is of rank $n-1$, since the null distribution is of rank 1. The converse is also true. Finally, $\mathcal{F}$ is degenerate if and only if each leaf of $\mathcal{F}$ is a degenerate hypersurface (cf. Bejancu [B96]). Thus summing up this discussion we may state the following.

Theorem 5.1. *Let $\mathcal{F}$ be a foliation of codimension one on an $(n+1)$–dimensional proper semi–Riemannian manifold (M, g). Then the following assertions are equivalent:*

(i) *$\mathcal{F}$ is a degenerate foliation.*
(ii) *$\mathcal{D}^{\perp}$ is a vector subbundle of $\mathcal{D}$.*
(iii) *The induced tensor field by g on $\mathcal{D}$ is of rank $n-1$.*
(iv) *Any leaf of $\mathcal{D}$ is a degenerate hypersurface.*

Hence, from now on $\mathcal{D}^{\perp}$ is a totally null distribution, that is, locally there exists a null vector field ξ such that $\mathcal{D}^{\perp} = \text{span}\{\xi\}$. We call ξ the **null structural vector field** of the degenerate foliation $\mathcal{F}$. Before we go further into the study let us present some examples of degenerate foliations.

Example 5.1. Let $\mathbb{R}^{n+1}_q = (\mathbb{R}^{n+1}, g)$ be the $(n+1)$–dimensional semi–Euclidean space with g given as in (1.4.9). Then consider $n+1$ fixed real numbers $\lambda_1, ..., \lambda_{n+1}$ satisfying

$$\sum_{t=1}^{q} (\lambda_t)^2 = \sum_{s=q+1}^{n+1} (\lambda_s)^2, \ (\lambda_1, ..., \lambda_{n+1}) \neq (0, ..., 0).$$

It is easy to see that the foliation by hyperplanes

$$\sum_{a=1}^{n+1} \lambda_a x^a = c, \quad c \in \mathbb{R},$$

is a degenerate foliation on $\mathbb{R}^{n+1}_q$ with null structural vector field

$$\xi = -\sum_{t=1}^{q} \lambda_t \frac{\partial}{\partial x^t} + \sum_{s=q+1}^{n+1} \lambda_s \frac{\partial}{\partial x^s}. \qquad \blacksquare$$

Example 5.2. Let $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, g)$ be the $(n+1)$–dimensional Lorentz space with g given as in (1.4.10). Denote by L the x^1–axis of $\mathbb{R}^{n+1}_1$ and consider the open submanifold $M = \mathbb{R}_1^{n+1} \backslash \{L\}$ of $\mathbb{R}_1^{n+1}$. Then denote by $\mathcal{F}^+$ and $\mathcal{F}^-$ the foliations on M with leaves given by

$$x^1 = \left(\sum_{s=2}^{n+1} (x^s)^2 \right)^{1/2} + c, \quad c \in \mathbb{R},$$

and

$$x^1 = - \left(\sum_{s=2}^{n+1} (x^s)^2 \right)^{1/2} + c, \quad c \in \mathbb{R},$$

respectively. Both $\mathcal{F}^+$ and $\mathcal{F}^-$ are degenerate foliations on M with null structural vector fields

$$\xi^+ = \frac{\partial}{\partial x^1} + \frac{1}{\alpha} \sum_{s=2}^{n+1} x^s \frac{\partial}{\partial x^s},$$

and

$$\xi^- = \frac{\partial}{\partial x^1} - \frac{1}{\alpha} \sum_{s=2}^{n+1} x^s \frac{\partial}{\partial x^s},$$

respectively, where we set

$$\alpha = \left(\sum_{s=2}^{n+1} (x^s)^2 \right)^{1/2}.$$

According to the terminology in physics under which leaves for $c = 0$ are known, we call $\mathcal{F}^+$ and $\mathcal{F}^-$ the **future cones foliation** and the **past cones foliation** respectively. ■

Example 5.3. Let M be the hypersurface of $\mathbb{R}_1^{n+1}$ situated in the half space $x^{n+1} > 0$ and given by the equation

$$\sum_{s=3}^{n+1} (x^s)^2 = 1.$$

Consider the distribution $\mathcal{D}$ on M spanned by the vector fields

$$X_2 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \quad X_s = \frac{\partial}{\partial x^s} - \frac{x^s}{x^{n+1}} \frac{\partial}{\partial x^{n+1}}, \quad s \in \{3, ..., n\}.$$

It is easy to check that $\mathcal{D}$ is an integrable distribution and $\mathcal{D}^\perp$ is spanned by $\xi = X_2$. Therefore $\mathcal{D}^\perp$ is a vector subbundle of $\mathcal{D}$, and by (ii) of Theorem 5.1 we conclude that $\mathcal{D}$ defines a degenerate foliation on M. ■

According to the general theory of degenerate distributions developed in Section 1.8 we may state the following (see Theorems 1.8.2 and 1.8.4).

Theorem 5.2.

(i) *Let $\mathcal{F}$ be a totally-null foliation on a 2-dimensional Lorentz manifold (M, g). Then there exists a unique totally-null distribution $\mathcal{D}'$ that is complementary to $\mathcal{D}$ in TM.*

(ii) *Let $\mathcal{F}$ be a degenerate n–foliation on an $(n+1)$–dimensional semi–Riemannian manifold (M, g) with $n > 1$. Then for a screen distribution $\mathcal{S}$ on M there exists a unique totally-null distribution $\mathcal{D}'(\mathcal{S})$ that is complementary to $\mathcal{D}$ in TM.*

As the case $n = 1$ was fully analyzed in Section 1.8 we concentrate only on the case $n > 1$. The second fundamental form B of the degenerate distribution $\mathcal{D}$ (cf. (1.8.19)) is also called **second fundamental form** of the degenerate foliation $\mathcal{F}$. B is a degenerate $F(M)$–bilinear form on $\Gamma(\mathcal{D})$, and does not depend on the screen distribution $\mathcal{S}$ on M. As in case of non–degenerate foliations we say that $\mathcal{F}$ is **totally geodesic** if B vanishes identically on M. Also, we say that $\mathcal{F}$ is **totally umbilical** if on each coordinate neighbourhood $\mathcal{U} \subset M$ there exists a smooth function ρ such that

$$B(X, Y) = \rho g(X, Y), \quad \forall\, X, Y \in \Gamma(\mathcal{D}_{|\mathcal{U}}). \tag{5.1}$$

It is easy to see that the foliation from Example 5.1 is totally geodesic because its leaves are degenerate hyperplanes which are totally geodesic immersed in $\mathbb{R}^{n+1}_q$ (cf. Bejancu [B96]). Now, we consider the foliation $\mathcal{F}^+$ from Example 5.2. Then the distribution $\mathcal{D}$ is spanned by the vector fields

$$X_s = \frac{\partial}{\partial x^s} + \frac{x^s}{\alpha}\,\frac{\partial}{\partial x^1}\,, \quad s \in \{2, ..., n+1\}.$$

By direct calculations using (1.8.19) and (1.8.15a) we obtain

$$B(X_s, X_r) = g(\widetilde{\nabla}_{X_s} X_r, \xi^+) = \frac{1}{\alpha^3}\,(x^s x^r - \alpha^2 \delta_{sr}).$$

Also, by (1.4.10) we have

$$g(X_s, X_r) = \frac{1}{\alpha^2}\,(\alpha^2 \delta_{sr} - x^r x^s).$$

Thus the future cones foliation $\mathcal{F}^+$ is totally umbilical with $\rho = -\dfrac{1}{\alpha}\cdot$ Similarly, it follows that $\mathcal{F}^-$ is also totally umbilical with the same function ρ.

Theorem 5.3. (Bejancu–Farran [BF03b]). *Let (M, g) be a 3–dimensional Lorentz manifold. Then any degenerate foliation of codimension one is either totally geodesic or totally umbilical.*

Proof. Suppose that locally $\mathcal{D} = \text{span}\{E, \xi\}$ where ξ spans $\mathcal{D}^\perp$ and E is a non–null vector field. Then by (1.8.25) we have

$$B(\xi,\xi) = B(E,\xi) = 0.$$

As $g(\xi,\xi) = g(E,\xi) = 0$, we see that (5.1) is satisfied with

$$\rho = h(E,E)/g(E,E).$$

Hence the foliation is either totally geodesic or totally umbilical, depending on whether $h(E,E) = 0$ or $h(E,E) \neq 0$, respectively. ∎

Now, according to the terminology in Section 1.5, ξ is $\mathcal{D}$–Killing if and only if

$$\begin{aligned}(\mathcal{L}_\xi g)(Y,Z) = \xi(g(Y,Z)) - g([\xi,Y],Z)\\ -g([\xi,Z],Y) = 0, \quad \forall\, Y,Z \in \Gamma(\mathcal{D}),\end{aligned} \tag{5.2}$$

where $\mathcal{L}$ is the Lie derivative on M. It is easy to see that (5.2) should be verified only for $Y,Z \in \Gamma(\mathcal{S})$, where $\mathcal{S}$ is a screen distribution for $\mathcal{D}^\perp$. Thus comparing (5.2) with (3.2) we may say that the degenerate metric g on $\mathcal{D}$ is **bundle–like** for the foliation $\mathcal{F}^\perp$ determined by $\mathcal{D}^\perp$. Next, we consider the second fundamental form B of $\mathcal{F}$ (cf. (1.8.19) and (1.8.15a))

$$B(Y,Z) = g(\widetilde{\nabla}_Y Z,\xi), \quad \forall\, Y,Z \in \Gamma(\mathcal{D}). \tag{5.3}$$

Then by using (1.5.10) and taking into account that $g(X,\xi) = 0$ for any $X \in \Gamma(\mathcal{D})$, we obtain

$$B(Y,Z) = -\frac{1}{2}\left\{\xi(g(Y,Z)) - g([\xi,Y],Z) - g([\xi,Z],Y)\right\}. \tag{5.4}$$

Comparing (5.4) with (5.2) we deduce an interesting characterization of totally geodesic degenerate foliations of codimension one.

Theorem 5.4. *Let $\mathcal{F}$ be a degenerate foliation of codimension one on (M,g) and $\mathcal{F}^\perp$ be the totally–null foliation determined by $\mathcal{D}^\perp$. Then $\mathcal{F}$ is totally geodesic if and only if g is bundle–like for $\mathcal{F}^\perp$.*

We should note that $\mathcal{D}^\perp \subset \mathcal{D}$, so we may say that $\mathcal{F}^\perp$ is a subfoliation of $\mathcal{F}$. However by the above result we can see that $\mathcal{F}^\perp$ gives a lot of information about the ambient foliation $\mathcal{F}$.

4

PARALLEL FOLIATIONS

This chapter is dedicated to studying the geometry of parallel foliations on semi–Riemannian manifolds. These are foliations whose tangent distributions are invariant under parallel transport with respect to the Levi–Civita connection. The way these distributions behave with respect to the semi–Riemannian metric is crucial and plays a major role in determining the geometry of both the foliations and the ambient manifolds. Although, the case when a tangent distribution is non–degenerate is very well determined, the situation for the degenerate case is still very far from being understood.

Our aim is to give a fairly comprehensive picture of what is known (or at least of what we know) about the geometry of a semi–Riemannian manifold on which a parallel foliation is defined. In the degenerate case this problem was completely solved as far as a local structure is concerned by A.G. Walker in the fifties of the last century. A definitive global structure theorem for the Riemannian case was obtained few years earlier by de Rham. The theorem of de Rham was extended by Wu to include the non–degenerate semi–Riemannian case, but as mentioned above, the global structure in the degenerate case has not been settled yet. We hope that geometers will be encouraged by this exposition to tackle the remaining unsolved problems.

The first section introduces the notion of parallelism in general, while the second discusses parallelism on almost product manifolds. In the third we move to parallelism with respect to the Levi–Civita connection on a semi–Riemannian manifold. Section 4.4 treats the non–degenerate case culminating with the most general form of the de Rham decomposition theorem.

Walker's results lie on the heart of the remaining sections. These sections were also greatly influenced by the way Walker's results were exploited by Robertson and Furness. The totally–null case was treated in Sections 4.5 and 4.6. The partially–null case is the most complicated and less understood one. It was visited briefly in Sections 4.7 and 4.8. Section 4.8 also treats the situation when the largest parallel degenerate foliation has a complementary foliation. The last section embarks on a very important notion in differential geometry, namely that of G–structures. The purpose of the section is to study parallel foliations on semi–Riemannian manifolds by using the theory of G–structures.

4.1 Parallelism

Let ∇ be a linear connection on a smooth m–dimensional manifold M. Recall that the tangent bundle TM has a natural m–foliation by fibers (see Example 2.1.4). The distribution VTM tangent to this foliation is known as the **vertical distribution** on TM. Geometrically, the linear connection ∇ assigns an m–distribution HTM on TM complementary to VTM as follows. Let (x^a, y^a) be a coordinate system on TM, where (x^a), $a \in \{1, ..., m\}$ are local coordinates on M. Then we put

$$\nabla_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^a} = \Gamma_a{}^c{}_b(x) \, \frac{\partial}{\partial x^c} \,, \tag{1.1}$$

and consider the functions

$$H_b^c(x, y) = y^a \, \Gamma_a{}^c{}_b(x). \tag{1.2}$$

Taking into account that $\{\Gamma_a{}^c{}_b(x)\}$ are the local coefficients of a linear connection on M, we define HTM as the distribution that is locally spanned by

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - H_a^b(x, y) \, \frac{\partial}{\partial y^b} \,, \;\; a \in \{1, ..., m\}. \tag{1.3}$$

From now on, HTM is called the **horizontal distribution** on TM induced by ∇.

A path $\sigma^* : [0, 1] \longrightarrow TM$ is said to be **horizontal** if $\dfrac{d\sigma^*}{dt} \in HTM_{\sigma^*(t)}$, for all $t \in [0, 1]$. Now, if $\sigma : [0, 1] \longrightarrow M$ is a piecewise smooth path in M taking $x = \sigma(0)$ to $y = \sigma(1)$ in M, then for each $u \in T_xM$, there is a unique horizontal lift $\sigma^* : [0, 1] \longrightarrow TM$ with $\sigma^*(0) = u$. This says that σ^* is horizontal and that $\pi(\sigma^*(t)) = \sigma(t)$ where $\pi : TM \longrightarrow M$ is the natural projection. Then it is easy to check that for any $t \in [0, 1]$

$$\tau_{\sigma(t)} : T_xM \longrightarrow T_{\sigma(t)}M, \quad \tau_{\sigma(t)}(u) = \sigma^*(t),$$

is an isomorphism of vector spaces. $\tau_{\sigma(t)}$ is known as the **parallel displacement** or **parallel transport** along σ. If in particular, M carries a semi–Riemannian metric g and ∇ is the Levi–Civita connection with respect to g, then $\tau_{\sigma(t)}$ is a linear isometry (cf. O'Neill [O83], p.66).

Conversely, given a distribution HTM complementary to VTM, the parallel displacement can be used to define covariant differentiation. This is done as follows. Let X and Y be two vector fields on M. For any point $x \in M$ we take the integral curve $\sigma : [0, 1] \longrightarrow M$ of X through x, that is, $\sigma(0) = x$ and

$\sigma'(t) = X(\sigma(t))$. Then the covariant derivative $\nabla_X Y$ of Y with respect to X is the vector field given by

$$(\nabla_X Y)(x) = \lim_{t \to 0} \frac{1}{t} \left(\tau^{-1}_{\sigma(t)} Y(\sigma(t)) - Y(x) \right). \tag{1.4}$$

Now, if $x \in M$, and $\sigma : [0, 1] \longrightarrow M$ is a loop at x (that is $\sigma(0) = \sigma(1) = x$), then the parallel displacement $\tau_{\sigma(t)}$ is an automorphism of $T_x M$. All such automorphisms define a group Φ_x known as the **holonomy group** of the connection ∇ at x. Since M is supposed to be connected, then holonomy groups at different points are isomorphic to each other and we can speak of the holonomy group Φ of the connection ∇. If the action of Φ_x on $T_x M$ leaves a non–trivial k–dimensional subspace $\mathcal{D}_x$ of $T_x M$ invariant, then Φ is said to be k–**reducible**. Accordingly, if Φ is k–reducible then we say that M is ∇–**reducible**. Otherwise, M is called ∇–**irreducible**.

We are now in a position to define parallel distributions on manifolds. So, let ∇ be a linear connection on an $(n+p)$–dimensional manifold M with $n > 0$, $p > 0$. An n–distribution $\mathcal{D}$ on M is said to be **parallel** with respect to ∇ if $\mathcal{D}$ is invariant under parallel displacements. That is to say, for all $x, y \in M$ and all piecewise smooth paths σ from x to y we have $\tau_\sigma(\mathcal{D}_x) = \mathcal{D}_y$.

Theorem 1.1. *Let ∇ be a linear connection on a connected smooth $(n+p)$–dimensional manifold M with $n > 0$, $p > 0$. Then M admits an n–distribution $\mathcal{D}$ parallel with respect to ∇ if and only if Φ is n–reducible.*

Proof. First, if $\mathcal{D}$ is a parallel n–distribution on M, then for any $x \in M$, and any loop σ at x we have $\tau_\sigma(\mathcal{D}_x) = \mathcal{D}_x$, and hence Φ is n–reducible. Conversely, suppose that Φ is n–reducible. Then for some $x \in M$ we take $\mathcal{D}_x$ to be the subspace invariant under Φ_x. Now we define a distribution $\mathcal{D}$ on M as follows. For any other point $y \in M$ we take $\mathcal{D}_y$ to be the image of $\mathcal{D}_x$ under any parallel displacement τ_σ from $T_x M$ to $T_y M$. To show that $\mathcal{D}_y$ is independent of the choice of σ, we consider any other path δ taking x to y. Then $\delta^{-1} \circ \sigma$ is a loop at x and hence $\mathcal{D}_x$ is invariant under the parallel displacement $\tau_{\delta^{-1} \circ \sigma} = \tau_\delta^{-1} \circ \tau_\sigma$. Thus $\tau_\delta^{-1} \circ \tau_\sigma(\mathcal{D}_x) = \mathcal{D}_x$, and hence $\tau_\sigma(\mathcal{D}_x) = \tau_\delta(\mathcal{D}_x)$. Thus $\mathcal{D}$ is well defined on M. The smoothness and parallelism of $\mathcal{D}$ follow directly from its construction. ■

Given a linear connection ∇ on M, the above theorem discusses the existence problem for a distribution $\mathcal{D}$ that is parallel with respect to ∇. The converse problem is to start with a distribution $\mathcal{D}$ on M and discuss the existence of a linear connection ∇ on M with respect to which $\mathcal{D}$ is parallel. Before we discuss this issue, we state a proposition whose proof follows directly by using (1.4).

Proposition 1.2. *Let ∇ be a linear connection and $\mathcal{D}$ a distribution on a manifold M. Then $\mathcal{D}$ is parallel with respect to ∇ if and only if ∇ is an adapted connection to $\mathcal{D}$.*

Now, we state the following.

Proposition 1.3. *Let $\mathcal{D}$ be a distribution on a paracompact manifold M. Then there is a linear connection on M with respect to which $\mathcal{D}$ is parallel.*

Proof. Since M is paracompact, it admits a Riemannian metric g and hence a Levi–Civita connection $\widetilde{\nabla}$ (see Corollary 1.5.2). The connection we are looking for is nothing but the Vrănceanu connection defined by $\widetilde{\nabla}$ (see (3.1.12)). ∎

In what follows we show that the integrability of a distribution is closely related to the torsion of a linear connection. First, we prove the following.

Proposition 1.4. *Let ∇ be a linear connection and $\mathcal{D}$ a distribution on a manifold M. If ∇ is torsion–free and $\mathcal{D}$ is parallel with respect to ∇, then $\mathcal{D}$ is integrable.*

Proof. Using Theorem 2.1.7, it is enough to show that $\mathcal{D}$ is involutive. Taking into account that ∇ is torsion–free, we have

$$[X,Y] = \nabla_X Y - \nabla_Y X, \quad \forall\, X,Y \in \Gamma(TM).$$

Then, by using Proposition 1.2, we deduce that $[X,Y] \in \Gamma(\mathcal{D})$, for any $X,Y \in \Gamma(\mathcal{D})$. Hence $\mathcal{D}$ is involutive. ∎

Next, by using the Vrănceanu connection, we prove the converse of the above proposition.

Proposition 1.5. *Let $\mathcal{D}$ be an integrable distribution on a paracompact manifold M. Then there exists a torsion–free linear connection ∇ on M such that $\mathcal{D}$ is parallel with respect to ∇.*

Proof. Let g be a Riemannian metric on M and ∇^* be the Vrănceanu connection defined by the Levi–Civita connection $\widetilde{\nabla}$ on (M,g). Since the second fundamental form h of $\mathcal{D}$ is symmetric (see the assertion (iii) of Lemma 1.5.5), from (1.6.14) we deduce that the torsion tensor field T^* of ∇^* is given by

$$T^*(X,Y) = h'(Q'Y,Q'X) - h'(Q'X,Q'Y), \quad \forall\, X,Y \in \Gamma(TM).$$

Then, by (1.5.21b) we obtain

$$T^*(X,Y) = Q\widetilde{\nabla}_{Q'Y}Q'X - Q\widetilde{\nabla}_{Q'X}Q'Y = -Q[Q'X,Q'Y], \tag{1.5}$$

since $\widetilde{\nabla}$ is torsion–free. Now, we define a new linear connection

$$\overline{\nabla}_X Y = \nabla^*_X Y - \frac{1}{2}\, T^*(X,Y), \quad \forall\, X,Y \in \Gamma(TM). \tag{1.6}$$

By using (1.5) and taking into account that $\mathcal{D}$ is parallel with respect to ∇^*, we conclude that $\mathcal{D}$ is parallel with respect to $\overline{\nabla}$ too. Finally, by direct calculations using (1.6) we deduce that $\overline{\nabla}$ is a torsion–free linear connection on M. This completes the proof of the proposition. ∎

Next, by combining Propositions 1.4 and 1.5, we can state the following.

Theorem 1.6. (Willmore [Wil56], Walker [Wal55], [Wal58]). *A distribution $\mathcal{D}$ on a manifold M is integrable if and only if there exists a torsion–free linear connection ∇ on M such that $\mathcal{D}$ is parallel with respect to ∇.*

In fact, Walker [Wal55] has studied the integrability and parallelism of a complete system of distributions in relation to the torsion of a linear connection. A family of r distributions $\mathcal{D}_1, ..., \mathcal{D}_r$ on M is said to be a **complete system of distributions** if $\mathcal{D}_i \cap \mathcal{D}_j = \{0\}$ for any $i \neq j$, and $\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_r = TM$. Since the only complete system of interest to us is composed of two complementary distributions, we only prove the following. (Technically, if the number of distributions is more than two, the proof is essentially similar.)

Theorem 1.7. *Let $(\mathcal{D}, \mathcal{D}')$ be a pair of complementary distributions on a manifold M. Then we have the assertions:*

(i) *There exists a linear connection ∇ on M such that both $\mathcal{D}$ and $\mathcal{D}'$ are parallel distributions with respect to ∇.*
(ii) *$\mathcal{D}$ and $\mathcal{D}'$ are both integrable if and only if there exists a torsion–free linear connection ∇^* on M such that $\mathcal{D}$ and $\mathcal{D}'$ are parallel with respect to ∇^*.*

Proof. Clearly, the Vrănceanu and Schouten–Van Kampen connections defined by the Levi–Civita connection on M with respect to a Riemannian metric have the property required in (i). The assertion (ii) is a consequence of Theorem 1.3.3. ∎

In general, a foliation $\mathcal{F}$ on a manifold M is said to be **parallel** with respect to a linear connection ∇, if the tangent distribution $\mathcal{D}$ of $\mathcal{F}$ is parallel with respect to ∇. Then from Proposition 1.5 and the assertion (ii) of Theorem 1.7 we deduce the following.

Corollary 1.8.

(i) *For any foliation $\mathcal{F}$ there exists a torsion–free linear connection ∇ on M such that $\mathcal{F}$ is parallel with respect to ∇.*
(ii) *For any two complementary foliations $\mathcal{F}$ and $\mathcal{F}'$ on M there exists a torsion–free linear connection ∇ on M such that both $\mathcal{F}$ and $\mathcal{F}'$ are parallel with respect to ∇.*

Before we end this section on parallelism, it is worth describing two weaker notions of parallelism that deserve some attention. These are the notions of relative parallelism and self–parallelism that we describe below.

Let $\mathcal{H}$ be a distribution on a manifold M. A path $\sigma : [0,1] \longrightarrow M$ is said to be **tangent** to $\mathcal{H}$ (or an **integral path** of $\mathcal{H}$) if for all $t \in [0,1]$ we have $\frac{d\sigma}{dt} \in \mathcal{H}_{\sigma(t)}$. In particular, if $\mathcal{H}$ is integrable, then the integral paths of $\mathcal{H}$ are just paths in the leaves of the foliation determined by $\mathcal{H}$. Now, suppose that $\mathcal{D}$ is another distribution on M (not necessarily distinct from $\mathcal{H}$), and ∇ a linear connection on M. We say that $\mathcal{D}$ is ∇–**parallel relative** to $\mathcal{H}$ if $\mathcal{D}$ is invariant under parallel displacements τ_σ for all paths σ tangent to $\mathcal{H}$. When $\mathcal{D}$ is ∇–parallel relative to itself, then it is called **self–parallel**. The next proposition follows directly from (1.4).

Proposition 1.9.

(i) *$\mathcal{D}$ is ∇–parallel relative to $\mathcal{H}$ if and only if $\nabla_X Y \in \Gamma(\mathcal{D})$ for any $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(\mathcal{D})$.*
(ii) *$\mathcal{D}$ is self–parallel if and only if $\nabla_X Y \in \Gamma(\mathcal{D})$ for any $X, Y \in \Gamma(\mathcal{D})$.*

It is interesting to note that in case of torsion–free linear connections the self–parallelism implies the parallelism, as it is stated below.

Proposition 1.10. *Let $\mathcal{D}$ be a self–parallel distribution with respect to a torsion–free linear connection ∇. Then $\mathcal{D}$ is parallel with respect to a torsion–free linear connection ∇'.*

Proof. Since ∇ is torsion–free, $\mathcal{D}$ is integrable. Then apply Proposition 1.5 and obtain the assertion. ■

4.2 Parallelism on Almost Product Manifolds

Let $\mathcal{D}$ be an n–distribution on an $(n+p)$–dimensional manifold M. In Section 1.1 we saw that we can always find a p–distribution $\mathcal{D}'$ complementary to $\mathcal{D}$, thus obtaining an almost product structure F on M given by (1.1.11). As usual, for the almost product manifold $(M, \mathcal{D}, \mathcal{D}')$ we keep the notations Q and Q' for the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then using Theorem 1.2.2 and Proposition 1.2 we obtain the following.

Theorem 2.1. *Let ∇^* be a linear connection on an almost product manifold $(M, \mathcal{D}, \mathcal{D}')$. Then the following assertions are equivalent:*

(i) *Both $\mathcal{D}$ and $\mathcal{D}'$ are parallel with respect to ∇^*.*
(ii) *F is parallel with respect to ∇^*.*
(iii) *Both Q and Q' are parallel with respect to ∇^*.*

If in particular ∇^* is a torsion–free linear connection, then by assertion (ii) of Theorem 1.7 we conclude that any of the assertions in Theorem 2.1 implies the integrability of both distributions $\mathcal{D}$ and $\mathcal{D}'$. Thus M is endowed with two complementary foliations $\mathcal{F}$ and $\mathcal{F}'$. Robertson [Rob70] called such a pair a ∇^*–**grid**, and the structure $(M, \nabla^*, \mathcal{F}, \mathcal{F}')$, a **grid manifold**.

To study the global geometry of almost product manifolds satisfying certain integrability and parallelism conditions we need the following models. Let N and N' be two smooth manifolds and $M = N \times N'$ be their product manifold. Then M carries two complementary foliations $\mathcal{F}$ and $\mathcal{F}'$ by copies of N and N' respectively. Let $\mathcal{D}$ and $\mathcal{D}'$ be their tangent distributions with projection morphisms Q and Q'. Now, if ∇ and ∇' are linear connections on N and N' respectively, then we can define a linear connection ∇^* on M as follows. For any point $x^* = (x, x')$ of M consider the coordinate systems $(x^1, ..., x^n; \mathcal{U})$ and $(x^{n+1}, ..., x^{n+p}; \mathcal{U}')$ about $x \in N$ and $x' \in N'$ respectively. Then $(x^1, ..., x^n, x^{n+1}, ..., x^{n+p}; \mathcal{U} \times \mathcal{U}')$ is a coordinate system about x^*. Suppose $\{\Gamma_j{}^i{}_k\}$, $i, j, k \in \{1, ..., n\}$ and $\{\Gamma'_\beta{}^\alpha{}_\gamma\}$, $\alpha, \beta, \gamma \in \{n+1, ..., n+p\}$ are the local coefficients of ∇ and ∇' with respect to the coordinate systems $(x^i; \mathcal{U})$ and $(x^\alpha; \mathcal{U}')$ respectively. Then we define the local coefficients of ∇^* with respect to the coordinate system $(x^i, x^\alpha; \mathcal{U} \times \mathcal{U}')$ as follows:

$$\begin{aligned} &\text{(a) } \Gamma^*{}_j{}^i{}_k = \Gamma_j{}^i{}_k, \quad i, j, k \in \{1, ..., n\}, \\ &\text{(b) } \Gamma^*{}_\beta{}^\alpha{}_\gamma = \Gamma'{}_\beta{}^\alpha{}_\gamma, \quad \alpha, \beta, \gamma \in \{n+1, ..., n+p\}, \\ &\text{(c) } \Gamma^*{}_r{}^t{}_s = 0, \text{for all other triplets.} \end{aligned} \tag{2.1}$$

Also, we can express ∇^* by the following invariant form

$$\nabla^*_X Y = (\nabla_{QX} QY, \nabla'_{Q'X} Q'Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.2}$$

The pair (M, ∇^*) is called the **affine product** of (N, ∇) and (N', ∇').

Next, we consider the manifolds N and N' endowed with two semi–Riemannian metrics $g = [g_{ij}(x^k)]$, $i, j, k \in \{1, ..., n\}$ and $g' = [g'_{\alpha\beta}(x^\gamma)]$, $\alpha, \beta, \gamma \in \{n+1, ..., n+p\}$, respectively. Then we define the semi–Riemannian metric $\widetilde{g}$ on $M = N \times N'$ by the formula

$$\widetilde{g}(X, Y) = g(QX, QY) + g'(Q'X, Q'Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.3}$$

Locally, we put

$$\widetilde{g}_{ab} = \widetilde{g}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right), \quad a, b \in \{1, ..., n+p\},$$

where

$$\left\{\frac{\partial}{\partial x^a}\right\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\right\}, \ a \in \{1, ..., n+p\}, \ i \in \{1, ..., n\}, \ \alpha \in \{n+1, ..., n+p\},$$

is the natural field of frames on $\mathcal{U} \times \mathcal{U}'$. Then from (2.3) we deduce that

$$[\widetilde{g}_{ab}(x^c)] = \begin{bmatrix} g_{ij}(x^k) & 0 \\ 0 & g'_{\alpha\beta}(x^\gamma) \end{bmatrix}, \tag{2.4}$$

is the matrix of the local components of $\widetilde{g}$. The manifold $(M, \widetilde{g})$ is called the **semi–Riemannian product** of (N, g) and (N', g').

The above two types of products will serve as local models for foliated almost product manifolds.

Theorem 2.2. *Let $(M, \mathcal{D}, \mathcal{D}')$ be an almost product manifold. If both $\mathcal{D}$ and $\mathcal{D}'$ are integrable, then every point $x^* \in M$ has a neighbourhood $\mathcal{V}^* = \mathcal{V}\times\mathcal{V}'$, where $\mathcal{V}$ and $\mathcal{V}'$ are open submanifolds of leaves of $\mathcal{D}$ and $\mathcal{D}'$ through x^*.*

Proof. We assume that $\mathcal{D}$ and $\mathcal{D}'$ are integrable distributions of rank n and p respectively. Then we have two complementary foliations $\mathcal{F}$ and $\mathcal{F}'$ whose leaves are of dimensions n and p respectively. Take L and L' to be the leaves through x^* of $\mathcal{F}$ and $\mathcal{F}'$ respectively. Then there is a foliated chart $(\mathcal{U}, \varphi)$ about x^* with local coordinates $(x^1, ..., x^n, x^{n+1}, ..., x^{n+p})$ such that each plaque of $\mathcal{F}$ is given by the equations

$$x^{n+1} = c^{n+1}, ..., x^{n+p} = c^{n+p}.$$

Moreover, since x^* is the origin of the coordinate system, we may take $(x^1, ..., x^n, 0, ..., 0)$ as local coordinates on $\mathcal{U}\cap L$. Similarly, we take another foliated chart $(\mathcal{U}', \varphi')$ about x^* with respect to $\mathcal{F}'$ such that $(0, ..., 0, x^{n+1}, ..., x^{n+p})$ are local coordinates on $\mathcal{U}' \cap L'$. Then we choose the open neighbourhoods $\mathcal{V}$ and $\mathcal{V}'$ of x^* in L and L' such that $\mathcal{V}\times\mathcal{V}' \subset \mathcal{U} \cap \mathcal{U}'$. Thus $\mathcal{V}^* = \mathcal{V}\times\mathcal{V}'$ is the required neighbourhood of x^* in M. ∎

It is worth mentioning that we can take $(x^1, ..., x^n, x^{n+1}, ..., x^{n+p})$ as a coordinate system on $\mathcal{V}^*$ compatible with both foliations $\mathcal{F}$ and $\mathcal{F}'$. That is to say,

$$\mathcal{D} = \text{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\} \quad \text{and} \quad \mathcal{D}' = \text{span}\left\{\frac{\partial}{\partial x^{n+1}}, \dots, \frac{\partial}{\partial x^{n+p}}\right\}, \tag{2.5}$$

on $\mathcal{V}^*$.

The above theorem justifies the term **locally product manifold** for a manifold with two complementary integrable distributions as we have seen in Section 1.5.

Theorem 2.3. *Let $(M, \mathcal{D}, \mathcal{D}')$ be an almost product manifold, and ∇^* a torsion–free linear connection on M. If $\mathcal{D}$ and $\mathcal{D}'$ are parallel with respect to ∇^*, then for each $x^* \in M$ there is a neighbourhood $\mathcal{V}^* \subset M$ and two submanifolds $\mathcal{V}$ and $\mathcal{V}'$ of M admitting torsion–free linear connections ∇ and ∇' such that $(\mathcal{V}^*, \nabla^*)$ is the affine product of $(\mathcal{V}, \nabla)$ and $(\mathcal{V}', \nabla')$.*

Proof. By the assertion (ii) of Theorem 1.7 we deduce that both distributions $\mathcal{D}$ and $\mathcal{D}'$ are integrable. Thus we can apply Theorem 2.2 and obtain the local product $\mathcal{V}^* = \mathcal{V}\times\mathcal{V}'$, where $\mathcal{V}$ and $\mathcal{V}'$ are open submanifolds of the leaves L and L' respectively. Now, using Theorem 1.2.1 we infer that ∇^* induces two linear connections ∇ and ∇' on $\mathcal{D}$ and $\mathcal{D}'$ respectively, and we have (see (1.2.4))

$$\nabla^*_X Y = \nabla_X QY + \nabla'_X Q'Y, \quad \forall X, Y \in \Gamma(TM). \tag{2.6}$$

By using (2.5) and (2.6) we obtain

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \nabla^*_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}\,, \quad \nabla'_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\alpha} = \nabla^*_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\alpha}\,. \tag{2.7}$$

Thus ∇ and ∇' from (2.6) define two torsion–free linear connections on $\mathcal{V}$ and $\mathcal{V}'$ whose coefficients are related with coefficients of ∇^* on $\mathcal{V}^*$ by (2.1a) and (2.1b). Moreover, from (2.7) we deduce that

$$\Gamma^*{}_i{}^\alpha{}_j = \Gamma^*{}_\alpha{}^i{}_\beta = 0. \tag{2.8}$$

Next, since ∇^* is torsion–free, we have

$$\nabla^*_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^\alpha} = \nabla^*_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^i}\,.$$

As the two parts of this equality belong to complementary distributions, we conclude that they must be zero. Hence we have

$$\Gamma^*{}_\alpha{}^k{}_i = \Gamma^*{}_\alpha{}^\gamma{}_i = \Gamma^*{}_i{}^k{}_\alpha = \Gamma^*{}_i{}^\gamma{}_\alpha = 0. \tag{2.9}$$

Finally, (2.8) and (2.9) imply (2.1c), and therefore $(\mathcal{V}^*, \nabla^*)$ is an affine product of $(\mathcal{V}, \nabla)$ and $(\mathcal{V}', \nabla')$. ∎

A manifold satisfying the conditions of Theorem 2.3 is called a **locally affine product manifold**. It is clear that every locally affine product manifold is a locally product manifold. The relationship in the opposite direction is given by the following corollary.

Corollary 2.4. *Every locally product manifold M admits a linear connection ∇^* such that (M, ∇^*) is a locally affine product manifold.*

Proof. Suppose that $(M, \mathcal{D}, \mathcal{D}')$ is a locally product manifold, that is, $\mathcal{D}$ and $\mathcal{D}'$ are both integrable. Then by the assertion (ii) of Theorem 1.7 it follows that there exists a torsion–free linear connection ∇^* on M with respect to which $\mathcal{D}$ and $\mathcal{D}'$ are parallel. Hence by Theorem 2.3 $(M, \mathcal{D}, \mathcal{D}')$ is a locally affine product manifold with respect to ∇^*. ∎

Now, if in addition ∇^* is complete and M is simply connected, then $(M, \mathcal{D}, \mathcal{D}')$ from Theorem 2.3 is globally an affine product. To be more specific, we end this section by stating the following important result of Kashiwabara [Kas59].

Theorem 2.5. *Let ∇^* be a complete torsion–free linear connection on an almost product manifold $(M, \mathcal{D}, \mathcal{D}')$, where M is simply connected. If $\mathcal{D}$ and $\mathcal{D}'$ are parallel with respect to ∇^*, then there exist two manifolds L and L' admitting linear connections ∇ and ∇' such that (M, ∇^*) is the affine product of (L, ∇) and (L', ∇').*

In fact, the manifolds L and L' are two leaves through a point $x^* \in M$ of the foliations $\mathcal{F}$ and $\mathcal{F}'$ defined by $\mathcal{D}$ and $\mathcal{D}'$ respectively. The connections ∇ and ∇' are induced by ∇^* as we defined them in the proof of Theorem 2.3. However, the complete proof of the above theorem is too technical and will be omitted. It uses parallel transport along piecewise geodesic segments in L and L' to construct a covering map from $(L, \nabla) \times (L', \nabla')$ to (M, ∇^*). Then the result follows from the fact that M is simply connected.

4.3 Parallelism on Semi–Riemannian Manifolds

Let (M, g) be an m–dimensional semi–Riemannian manifold, and $\widetilde{\nabla}$ the Levi–Civita connection on M. As we have seen in Section 1.4, if $\mathcal{D}$ is a distribution on M, then using g we define the orthogonal distribution $\mathcal{D}^\perp$. Two more distributions arise in a natural way, namely, the distribution $\mathcal{D}^+ = \mathcal{D} + \mathcal{D}^\perp$ and $\mathcal{N} = \mathcal{D} \cap \mathcal{D}^\perp$. Notice that, in general, $\mathcal{D}^+$ need not be equal to TM and $\mathcal{N}$ need not be trivial, because this depends upon the degree of nullity of $\mathcal{D}$. Of course, if (M, g) is Riemannian, or in general when $\mathcal{D}$ is semi–Riemannian, then $\mathcal{D}^+ = TM$ and $\mathcal{N} = \{0\}$.

Now, we suppose that $\mathcal{N}, \mathcal{D}$ and $\mathcal{D}^\perp$ are distributions of rank r, $r+s$ and $r+u$ respectively. To determine the rank for $\mathcal{D}^+$, we recall the following result from linear algebra with respect to the dimensions of subspaces in a vector space (see O'Neill [O83], p. 49)

$$\dim \mathcal{D}^+_x = \dim \mathcal{D}_x + \dim \mathcal{D}^\perp_x - \dim \mathcal{N}_x, \quad \forall\, x \in M. \tag{3.1}$$

Hence $\mathcal{D}^+$ is a distribution of rank $r + s + u$. Moreover, by using (1.4.3) in (3.1) we deduce that $\mathcal{D}^+$ is of rank $m - r$. Hence the dimension of the manifold can be expressed as follows

$$m = 2r + s + u. \tag{3.2}$$

In order to stress the degree of nullity for each of the above distributions, we also say that $\mathcal{D}, \mathcal{D}^\perp, \mathcal{D}^+$ and $\mathcal{N}$ are of types (r, s), (r, u), $(r, s+u)$ and $(r, 0)$ respectively. Now, by using the terminology from Section 1.4 we see that $\mathcal{D}$ must be in exactly one of the following three classes:

a) $\mathcal{D}$ is non–degenerate (semi–Riemannian), if $r = 0,\ s > 0$.
b) $\mathcal{D}$ is partially–null, if $r > 0,\ s > 0$.
c) $\mathcal{D}$ is totally–null, if $r > 0,\ s = 0$.

Remark 3.1. Parallelism, in the rest of this chapter will be considered only with respect to the Levi–Civita connection on (M, g). ∎

Theorem 3.1. *Let $\mathcal{D}$ be a distribution that is parallel with respect to the Levi–Civita connection $\widetilde{\nabla}$ on (M, g). Then $\mathcal{D}^{\perp}, \mathcal{D}^{+}$ and $\mathcal{N}$ are also parallel with respect to $\widetilde{\nabla}$.*

Proof. First let us show that $\mathcal{D}^{\perp}$ is parallel with respect to $\widetilde{\nabla}$. Since g is parallel with respect to $\widetilde{\nabla}$, from (1.5.9) we deduce that

$$g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z) = 0, \quad \forall X \in \Gamma(TM),\ Y \in \Gamma(\mathcal{D}),\ Z \in \Gamma(\mathcal{D}^{\perp}).$$

As $\widetilde{\nabla}_X Y \in \Gamma(\mathcal{D})$, we have $g(\widetilde{\nabla}_X Y, Z) = 0$. Hence $g(Y, \widetilde{\nabla}_X Z) = 0$, which implies that $\widetilde{\nabla}_X Z \in \Gamma(\mathcal{D}^{\perp})$. Thus $\mathcal{D}^{\perp}$ is parallel with respect to $\widetilde{\nabla}$. Next, let $U \in \Gamma(\mathcal{D}^{+})$, that is, $U = Y + Z$, where $Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. Then by using the linearity of $\widetilde{\nabla}$ and the parallelism of both $\mathcal{D}$ and $\mathcal{D}^{\perp}$ we obtain

$$\widetilde{\nabla}_X U = \widetilde{\nabla}_X Y + \widetilde{\nabla}_X Z \in \Gamma(\mathcal{D}^{+}), \quad \forall X \in \Gamma(TM).$$

Hence $\mathcal{D}^{+}$ is parallel with respect to $\widetilde{\nabla}$. Finally, take $Y \in \Gamma(\mathcal{N})$ and $X \in \Gamma(TM)$. Then $Y \in \Gamma(\mathcal{D})$ and $Y \in \Gamma(\mathcal{D}^{\perp})$, which implies that $\widetilde{\nabla}_X Y \in \Gamma(\mathcal{D})$ and $\widetilde{\nabla}_X Y \in \Gamma(\mathcal{D}^{\perp})$. Hence $\widetilde{\nabla}_X Y \in \Gamma(\mathcal{N})$, that is, $\mathcal{N}$ is parallel with respect to $\widetilde{\nabla}$. ∎

Next, due to (3.2) we identify $\mathbb{R}^m$ with the product $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^r$, and denote points of $\mathbb{R}^m$ by the 4–tuples (x, y, z, t) accordingly. Then we have the following.

Theorem 3.2. *Let $\mathcal{D}$ be a parallel distribution of type (r, s) on the $(2r+s+u)$–dimensional semi–Riemannian manifold (M, g). Then M admits a foliated atlas $\mathcal{A}$ in which the coordinate transformations are given by*

$$\begin{aligned} \widetilde{x} &= \widetilde{x}(x, y, z, t), \ \widetilde{y} = \widetilde{y}(y, t), \\ \widetilde{z} &= \widetilde{z}(z, t), \qquad \widetilde{t} = \widetilde{t}(t). \end{aligned} \tag{3.3}$$

Furthermore, the distributions $\mathcal{N}, \mathcal{D}, \mathcal{D}^{\perp}$ and $\mathcal{D}^{+}$ are locally spanned by

$$\begin{aligned} &\text{(a)}\ \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r} \right\}, \\ &\text{(b)}\ \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^s} \right\}, \\ &\text{(c)}\ \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^u} \right\}, \\ &\text{(d)}\ \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^s}, \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^u} \right\}, \end{aligned} \tag{3.4}$$

respectively.

Proof. Using Theorem 3.1, we have altogether four distributions parallel with respect to the torsion–free linear connection $\widetilde{\nabla}$ on (M, g). Thus by Theorem 1.6, the distributions $\mathcal{N}, \mathcal{D}, \mathcal{D}^{\perp}$ and $\mathcal{D}^{+}$ define four parallel foliations $\mathcal{F}_{\mathcal{N}}$, $\mathcal{F}$, $\mathcal{F}^{\perp}$ and $\mathcal{F}^{+}$ respectively. Then the two assertions of the theorem follow from (2.1.5) and Theorem 1.1.1, taking into consideration that $\mathcal{F}_{\mathcal{N}}$ foliates every leaf of $\mathcal{F}$, $\mathcal{F}^{\perp}$ and $\mathcal{F}^{+}$, and that both $\mathcal{F}$ and $\mathcal{F}^{\perp}$ foliate every leaf of $\mathcal{F}^{+}$. ∎

A foliation $\mathcal{F}$ on a semi–Riemannian manifold (M, g) is said to be of type (r, s) if its tangent distribution $\mathcal{D}$ is of type (r, s). Thus $\mathcal{F}$ is **non–degenerate** (**partially–null**, **totally–null**) if $\mathcal{D}$ is so. When $r > 0$, $s > 0$, $u > 0$ it is easy to see that $\mathcal{F}_{\mathcal{N}}$ is totally–null, while the other three foliations are partially–null. We also note that in this study we have two flags of foliations:

$$\mathcal{F}_{\mathcal{N}} \subset \mathcal{F} \subset \mathcal{F}^{+} \text{ and } \mathcal{F}_{\mathcal{N}} \subset \mathcal{F}^{\perp} \subset \mathcal{F}^{+}.$$

In general, a flag of foliations is a family of foliations $\mathcal{F}_1, ..., \mathcal{F}_k$ of codimensions $q_1, ..., q_k$ $(q_1 \leq q_2 \leq \cdots \leq q_k)$ such that for $i < j$ the leaves of $\mathcal{F}_j$ are submanifolds of leaves of $\mathcal{F}_i$. Feigin [Fei75] introduced the concept of flag of foliations and developed a theory of its characteristic classes. In this respect, several results have been obtained for flags with two foliations, which are also called subfoliations (see Cordero [Cor85], Cordero–Gadea [CG76]).

The information we have from Theorem 3.2 will be used in what follows to study the geometry of semi–Riemannian manifolds admitting a parallel foliation $\mathcal{F}$. We must distinguish between the cases where $\mathcal{F}$ is non–degenerate, partially–null or totally–null.

4.4 Parallel Non–Degenerate Foliations

Let $\mathcal{F}$ be a parallel non–degenerate n–foliation on an $(n + p)$–dimensional semi–Riemannian manifold $(M, \widetilde{g})$. Thus the tangent distribution $\mathcal{D}$ to $\mathcal{F}$ is non–degenerate and parallel with respect to the Levi–Civita connection $\widetilde{\nabla}$ on $(M, \widetilde{g})$. Hence $\mathcal{D}^{\perp}$ is parallel, non–degenerate and complementary orthogonal to $\mathcal{D}$. This gives the second parallel p–foliation $\mathcal{F}^{\perp}$. Thus $(M, \mathcal{D}, \mathcal{D}^{\perp})$ is an almost product manifold and the pair $(\mathcal{F}, \mathcal{F}^{\perp})$ is a $\widetilde{\nabla}$–grid. Using Theorem 3.2 for $r = 0$ and Theorem 2.3 we obtain the following.

Theorem 4.1. *Let $\mathcal{F}$ be a parallel non–degenerate n–foliation on an $(n+p)$–dimensional semi–Riemannian manifold $(M, \widetilde{g})$. Then we have the assertions:*

(i) *For each $x \in M$ there is a coordinate neighbourhood $\mathcal{V}^*$ and two submanifolds $\mathcal{V}$ and $\mathcal{V}^{\perp}$ of M admitting torsion–free linear connections ∇ and $\nabla^{\perp}$ such that $(\mathcal{V}^*, \widetilde{\nabla})$ is the affine product of $(\mathcal{V}, \nabla)$ and $(\mathcal{V}^{\perp}, \nabla^{\perp})$.*

(ii) *If* $(x^1, ..., x^n, x^{n+1}, ..., x^{n+p})$ *are the coordinates on* $\mathcal{V}^*$, *then*

$$\mathcal{D} = \text{span}\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}, \quad \mathcal{D}^\perp = \text{span}\left\{\frac{\partial}{\partial x^{n+1}}, \ldots, \frac{\partial}{\partial x^{n+p}}\right\},$$

and the transformations of coordinates are given by

$$\widetilde{x}^i = \widetilde{x}^i(x^j), \quad \widetilde{x}^\alpha = \widetilde{x}^\alpha(x^\beta).$$

It is clear that with respect to the above coordinate system the matrix of the local components of $\widetilde{g}$ has the form

$$[\widetilde{g}_{ab}] = \begin{bmatrix} g_{ij}(x) & 0 \\ 0 & g_{\alpha\beta}(x) \end{bmatrix}, \quad a, b \in \{1, ..., n+p\},$$

where we set

$$g_{ij}(x) = \widetilde{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad i, j \in \{1, ..., n\},$$

and

$$g_{\alpha\beta}(x) = \widetilde{g}\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right), \quad \alpha, \beta \in \{n+1, ..., n+p\}.$$

Now, we want to show that the matrices $[g_{ij}(x)]$ and $[g_{\alpha\beta}(x)]$ define semi–Riemannian metrics g and $g^\perp$ on $\mathcal{V}$ and $\mathcal{V}^\perp$ respectively. Thus we must show that g_{ij} are independent of $(x^{n+1}, ..., x^{n+p})$ for all $i, j \in \{1, ..., n\}$ and $g_{\alpha\beta}$ are independent of $(x^1, ..., x^n)$ for all $\alpha, \beta \in \{n+1, ..., n+p\}$. First, from (2.2) written for $\widetilde{\nabla}, \nabla$ and $\nabla^\perp$ we deduce that $\widetilde{\nabla}_X Y = 0$ for any $X \in \Gamma(\mathcal{D}^\perp)$ and $Y \in \Gamma(\mathcal{D})$. Then since $\widetilde{g}$ is parallel with respect to $\widetilde{\nabla}$, we obtain

$$X(\widetilde{g}(Y, Z)) = \widetilde{g}(\widetilde{\nabla}_X Y, Z) + \widetilde{g}(Y, \widetilde{\nabla}_X Z) = 0,$$

for any $X \in \Gamma(\mathcal{D}^\perp)$ and $Y, Z \in \Gamma(\mathcal{D})$. Now, take $X = \frac{\partial}{\partial x^\alpha}$, $Y = \frac{\partial}{\partial x^i}$ and $Z = \frac{\partial}{\partial x^j}$, and obtain that g_{ij} are independent of $(x^{n+1}, ..., x^{n+p})$. Similarly, $[g_{\alpha\beta}]$ defines a semi–Riemannian metric on $\mathcal{V}^\perp$. Summing up, we have proved the following.

Theorem 4.2. *Let* $\mathcal{F}$ *be a parallel non–degenerate* n*–foliation on an* $(n+p)$*–dimensional semi–Riemannian manifold* $(M, \widetilde{g})$. *Then for any point* $x \in M$, *there is a neighbourhood* $\mathcal{V}^* \subset M$ *and two submanifolds* $\mathcal{V}$ *and* $\mathcal{V}^\perp$ *of dimensions* n *and* p, *admitting semi–Riemannian metrics* g *and* $g^\perp$ *such that* $(\mathcal{V}^*, \widetilde{g})$ *is the semi–Riemannian product of* $(\mathcal{V}, g)$ *and* $(\mathcal{V}^\perp, g^\perp)$.

From the above theorem we deduce that the matrix of the local components of $\widetilde{g}$ has the **canonical form**

$$[\widetilde{g}_{ab}] = \begin{bmatrix} g_{ij}(x^k) & 0 \\ 0 & g_{\alpha\beta}(x^\gamma) \end{bmatrix}, \tag{4.1}$$

where $a, b \in \{1, ..., n+p\}$, $i, j, k \in \{1, ..., n\}$, $\alpha, \beta, \gamma \in \{n+1, ..., n+p\}$.

Now, we characterize semi–Riemannian manifolds from the above theorem by using totally geodesic foliations studied in Section 3.4.

First, we note that a parallel non–degenerate foliation $\mathcal{F}$ on $(M, \widetilde{g})$ is totally geodesic since $\widetilde{\nabla}_X Y \in \Gamma(\mathcal{D})$ for any $X, Y \in \Gamma(\mathcal{D})$. However, the converse is not true. To show this we consider $M = \mathbb{R}^2 \backslash \{0\}$ endowed with the usual Euclidean metric $\widetilde{g}$ (see (1.4.11)). Then the connected components of the lines $ax+by = 0$ taken for all $(a, b) \neq (0, 0)$, determine a totally geodesic foliation on $(M, \widetilde{g})$ which is not parallel. The next theorem sheds more light on this problem.

Theorem 4.3. *Let $(M, \widetilde{g})$ be a semi–Riemannian manifold. Then the following assertions are equivalent:*

(i) *There exists a parallel non–degenerate foliation on M.*
(ii) *There exist two complementary orthogonal totally geodesic foliations on M.*

Proof. Let $\mathcal{F}$ be a parallel non–degenerate foliation on M and $\mathcal{D}$ its tangent distribution. Then for any $X, Y \in \Gamma(\mathcal{D})$ we have $\widetilde{\nabla}_X Y \in \Gamma(\mathcal{D})$. Thus, by (3.2.5) it follows that the second fundamental form h of $\mathcal{F}$ vanishes identically on M. Hence $\mathcal{F}$ is totally geodesic. Now, by Theorem 3.1, $\mathcal{D}^\perp$ is also parallel with respect to $\widetilde{\nabla}$ and therefore integrable. In a similar way as above, it follows that $\mathcal{F}^\perp$ is totally geodesic. Thus (i) implies (ii). Next, suppose $(\mathcal{F}, \mathcal{D})$ and $(\mathcal{F}^\perp, \mathcal{D}^\perp)$ are two complementary orthogonal totally geodesic foliations on M. Hence both are necessarily non–degenerate foliations. Now, by (3.2.5) and (3.2.6) we obtain

$$\widetilde{\nabla}_{QX} QY \in \Gamma(\mathcal{D}) \text{ and } \widetilde{\nabla}_{Q'X} Q'Y \in \Gamma(\mathcal{D}^\perp), \quad \forall\, X, Y \in \Gamma(TM).$$

Moreover, since $\widetilde{g}$ is parallel with respect to $\widetilde{\nabla}$, we have

$$\widetilde{g}(\widetilde{\nabla}_{Q'X} QY, Q'Z) = -g(QY, \widetilde{\nabla}_{Q'X} Q'Z) = 0.$$

Hence, $\widetilde{\nabla}_{Q'X} QY \in \Gamma(\mathcal{D})$, and thus $\mathcal{D}$ is parallel with respect to $\widetilde{\nabla}$. ■

The above equivalence can be used to get an elegant proof of the last part of the assertion in Theorem 4.2. Indeed, since we have two complementary totally geodesic non–degenerate foliations, their second fundamental forms vanish identically on M. Thus by assertion (vi) of Theorem 3.3.3 we deduce that both foliations are with bundle–like metric. Finally, by Theorem 3.3.2 we obtain that $[g_{ij}]$ and $[g_{\alpha\beta}]$ represent the matrices of two semi–Riemannian metrics on $\mathcal{V}$ and $\mathcal{V}^\perp$ respectively.

The Theorem 4.2 justifies the name **locally semi–Riemannian product** used in Section 1.5. Also we note that the manifold $(M, \widetilde{g})$ in this theorem does not have to be a global product as we can see from the following example.

Example 4.1. Consider the 2-dimensional torus $\mathbb{T}^2$ as the quotient space $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined using the action $(m, n)(x, y) = (x + m, y + n)$. Let θ be an irrational number, and $\mathcal{F}$ the parallel foliation of $\mathbb{R}^2$ whose leaves are straight lines of slope θ. This foliation is invariant under the action of $\mathbb{Z}^2$, which acts as a group of isometries of $\mathbb{R}^2$. So $\mathcal{F}$ induces a parallel foliation $\widetilde{\mathcal{F}}$ on the torus $\mathbb{T}^2$. Both foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^{\perp}$ have no compact leaves. Thus the product $L \times L^{\perp}$ of two leaves is not compact, and therefore cannot be diffeomorphic (not even homeomorphic) to the compact manifold $\mathbb{T}^2$. ∎

Now, let $(M, \widetilde{g})$ be a complete and simply connected semi–Riemannian manifold which admits a parallel non–degenerate foliation $\mathcal{F}$. Then using Theorem 2.5 and Theorem 4.2 one concludes that $(M, \widetilde{g})$ is a global semi–Riemannian product $(L, g) \times (L^{\perp}, g^{\perp})$, where L and $L^{\perp}$ are leaves of $\mathcal{F}$ and $\mathcal{F}^{\perp}$ through a point $x \in M$. To be more specific we give the following definition. Let (M, g) and $(\overline{M}, \bar{g})$ be two m–dimensional semi–Riemannian manifolds endowed with foliations $\mathcal{F}$ and $\overline{\mathcal{F}}$ respectively. Then an isometry $f : (M, g) \longrightarrow (\overline{M}, \bar{g})$ is called a **foliation preserving isometry** if it carries every leaf of $\mathcal{F}$ to a leaf of $\overline{\mathcal{F}}$. Now, we can state the following.

Theorem 4.4. *Let $(M, \widetilde{g})$ be a complete and simply connected semi–Riemannian manifold which admits a parallel non–degenerate foliation $\mathcal{F}$. Then there exists a foliation preserving isometry from $(M, \widetilde{g})$ onto the semi–Riemannian product $(L, g) \times (L^{\perp}, g^{\perp})$, where L and $L^{\perp}$ are the leaves of $\mathcal{F}$ and $\mathcal{F}^{\perp}$ through a point $x \in M$, and g and $g^{\perp}$ are the semi–Riemannian metrics induced by $\widetilde{g}$ on L and $L^{\perp}$ respectively.*

If the manifold is not simply connected, the following will be an immediate corollary.

Corollary 4.5. *Let $(M, \widetilde{g})$ be a complete semi–Riemannian manifold which admits a parallel non–degenerate foliation $\mathcal{F}$. Then there is a semi–Riemannian product $(M^*, g^*) = (\overline{L}, \bar{g}) \times (\overline{L}^{\perp}, \bar{g}^{\perp})$ and a properly discontinuous group G of isometries of (M^*, g^*) such that (M, g) is isometric to $(M^*, g^*)/G$. Furthermore, $\overline{L}$ and $\overline{L}^{\perp}$ are universal covering spaces of the leaves L and $L^{\perp}$ of $\mathcal{F}$ and $\mathcal{F}^{\perp}$ through a point $x \in M$, and G is isomorphic to $\Pi_1(M)$.*

Let us give here some history of studying the geometry of a semi–Riemannian manifold admitting a parallel non–degenerate foliation. The local product situation (Theorem 4.2) was first proved by Thomas [Tho39] in 1939 for Riemannian manifolds. The global product result (Theorem 4.4) was first proved by de Rham [deR52] in 1952, again in the Riemannian case only. Another proof of this theorem in the Riemannian case was given in Kobayashi–Nomizu [KN63], p. 187. This proof uses Reinhart's work [Rei59a] on foliations with bundle-like metric (see Section 3.3). A proof in the general situation of semi–Riemannian manifolds was first given by Wu [Wu64] in 1964. Wu used the holonomy theorem of Ambrose and Singer to convert the reducibility property into a statement about the behaviour of curvature under parallel displacement. Since parallel displacement of curvature determines M up to isometry, the local product decomposition is obtainable from a study of the curvature form, and the global structure is then deduced using the simple connectedness. The proof using Kashiwabara's result (Theorem 2.5) was given by Furness [Fur72] in 1972.

We cannot end this section without giving the general **de Rham Decomposition Theorem**. So, let $(M, \widetilde{g})$ be a Riemannian manifold and $\widetilde{\nabla}$ the Levi–Civita connection defined by $\widetilde{g}$. In what follows we suppose that M is $\widetilde{\nabla}$-reducible. Shortly, we say that M is **reducible**. If M admits a parallel foliation $\mathcal{F}$, then $\mathcal{F}$ is automatically non–degenerate. Then it might happen that $\mathcal{F}$ admits a parallel subfoliation $\mathcal{F}'$, and we can subject $\mathcal{F}'$ to the same scrutiny. Thus we can envisage a maximal decomposition of M into mutually orthogonal parallel foliations $\mathcal{F}_1, ..., \mathcal{F}_k$. This can be done precisely by looking again at the action of the holonomy group Φ_x (see Section 4.1) on T_xM with respect to $\widetilde{\nabla}$. First consider the set

$$T_x^0 = \{v \in T_xM : \tau(v) = v, \ \forall \tau \in \Phi_x\}.$$

That is, T_x^0 is the set of all fixed points of Φ_x. Then T_x^0 is a linear subspace of T_xM and its orthogonal complement $(T_x^0)^\perp$ in T_xM is also invariant under Φ_x. Thus $(T_x^0)^\perp$ may be decomposed into a direct sum $T_x^1 \oplus \cdots \oplus T_x^r$ of irreducible mutually orthogonal Φ_x–invariant subspaces of T_xM. The decomposition

$$T_xM = T_x^0 \oplus T_x^1 \oplus \cdots \oplus T_x^r,$$

is called the **canonical decomposition** of T_xM. Since M is supposed to be reducible, this decomposition is non–trivial, that is, it has at least two subspaces of T_xM. Now, it follows that parallel displacements of T_x^i, $i \in \{0, ..., r\}$, yield parallel distributions $\mathcal{D}^i$ that are mutually orthogonal. Each $\mathcal{D}^i$ is integrable and non–degenerate (since $(M, \widetilde{g})$ is Riemannian) thus giving a parallel non–degenerate foliation $\mathcal{F}^i$. The foliation $\mathcal{F}^0$ has the special feature that each of its leaves is locally Euclidean. Thus for each $x \in M$, the leaf L^0 through x is a flat Riemannian manifold, that is, L^0 admits, locally, a basis of s parallel vector fields, where $s = \dim L^0$ (cf. Besse [Be87], p. 283). Indeed, since the holonomy group of L^0 consists of the identity only, $T_x^0 = S_x^1 \oplus \cdots \oplus S_x^s$, where S_x^t, $t \in \{1, ..., s\}$ are Φ_x–invariant lines. Now, on a neighbourhood $\mathcal{V}^0$ in L^0 we consider the unit vector fields X^t that span the line distributions

$$S^t = \bigcup_{x \in \mathcal{V}^0} S_x^t, \ \ \forall t \in \{1, ..., s\}.$$

Finally, taking into account that S^t are parallel and X^t are unit vector fields, we deduce that $\widetilde{\nabla}_X X^t = 0$, for any $X \in \Gamma(T\mathcal{V}^0)$ and $t \in \{1, ..., s\}$. Thus $\{X^1, ..., X^s\}$ is the basis we were looking for.

Summing up the above discussion and taking into account that Theorem 4.2 is true for more than two distributions, we obtain the following.

Theorem 4.6. *Let $(M, \widetilde{g})$ be a reducible Riemannian manifold with the canonical decomposition*

$$TM = \mathcal{D}^0 \oplus \mathcal{D}^1 \oplus \cdots \oplus \mathcal{D}^r.$$

Then any point $x \in M$ has a neighbourhood $\mathcal{V}^* = \mathcal{V}^0 \times \mathcal{V}^1 \times \cdots \times \mathcal{V}^r$ such that $(\mathcal{V}^*, \widetilde{g})$ is the Riemannian product $(\mathcal{V}^0, g^0) \times (\mathcal{V}^1, g^1) \times \cdots \times (\mathcal{V}^r, g^r)$, where $\mathcal{V}^i$ are neighbourhoods in the leaves L^i of $\mathcal{D}^i$ through x, and g^i are the Riemannian metrics induced by $\widetilde{g}$ on $\mathcal{V}^i$, $i \in \{0, ..., r\}$. Moreover, any leaf of L^0 is locally Euclidean.

The foliation $\mathcal{F}^0$ is unique, and $\mathcal{F}^1, ..., \mathcal{F}^r$ are unique up to order. This follows from the corresponding uniqueness properties of the canonical decomposition. Finally, by using Theorem 4.4 for more than two foliations and Theorem 4.6, we obtain the following general version of the de Rham Decomposition Theorem.

Theorem 4.7. *A complete, simply connected and reducible Riemannian manifold $(M, \widetilde{g})$ is isometric to the Riemannian product $(L^0, g^0) \times (L^1, g^1) \times \cdots \times (L^r, g^r)$, where (L^0, g^0) is a Euclidean space (possibly of dimension 0) and (L^i, g^i), $i \in \{1, ..., r\}$ are complete, simply connected and irreducible Riemannian manifolds. This decomposition is unique up to an order.*

If $(M, \widetilde{g})$ is not simply connected, then as in Corollary 4.5 it is isometric to the quotient space of such a Riemannian product under the action of a proper discontinuous group G that is isomorphic to $\Pi_1(M)$.

The case in which an m–dimensional semi–Riemannian manifold $(M, \widetilde{g})$ has a parallel non–degenerate 1–foliation $\mathcal{F}$, has some special features of interest. By Corollary 4.5, $(M, \widetilde{g})$ is universally covered by $\widetilde{M} = \mathbb{R} \times N$, where N is some simply connected $(m-1)$–dimensional manifold. This suggests a way of constructing semi–Riemannian manifolds admitting parallel non–degenerate foliations, using the technique of suspending a diffeomorphism as follows. Let N be a smooth n–dimensional manifold, where $n = m-1$, and let $f : N \longrightarrow N$ be a diffeomorphism. Take $\widetilde{M} = \mathbb{R} \times N = \{(t, x) : t \in \mathbb{R},\ x \in N\}$, and define an action of the additive group $\mathbb{Z}$ of integers on $\widetilde{M}$ by

$$\Phi_i(t, x) = (t + i, f^i(x)), \quad \forall\, i \in \mathbb{Z},\ (t, x) \in \mathbb{R} \times N.$$

Then $M = \widetilde{M}/\mathbb{Z}$ is an m–dimensional manifold, and is said to be the manifold obtained by the **suspension** of f. But $\widetilde{M}$ being a global product, it has a pair of complementary foliations: a 1–foliation $\widetilde{\mathcal{F}}$ given by $x =$ constant and an $(m-1)$–foliation $\widetilde{\mathcal{F}}'$ given by $t =$ constant. The action of $\mathbb{Z}$, as defined above, preserves both of the foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$, thus inducing a pair $\mathcal{F}, \mathcal{F}'$ of complementary foliations on M, where $\mathcal{F}$ is of dimension 1. Now, suppose that N has a semi–Riemannian metric h for which f is an isometry, and let e be the standard Euclidean metric on the real line $\mathbb{R}$. Then $\widetilde{g} = e \times h$ is a semi–Riemannian metric on $\widetilde{M}$ and $\mathbb{Z}$ acts on $(\widetilde{M}, \widetilde{g})$ as a group of isometries preserving the product structure. Thus $\widetilde{g}$ projects to define a metric g on M with respect to which $\mathcal{F}$ is parallel and non–degenerate. It is worth mentioning that $\mathcal{F}'$, as well, is parallel and non–degenerate with respect to g. Conversely, if there is a metric g on M such that $\mathcal{F}$ and $\mathcal{F}'$ are parallel, non–degenerate and mutually orthogonal, then there is a unique metric $\widetilde{g}$ on $\widetilde{M}$ such that the covering is Riemannian (see Wolf [Wol67]). Since g is locally the product of two metrics, then $\widetilde{g}$ is locally the product of two metrics, one is on $\mathbb{R}$, the second is on N. Let us denote this second metric by h. Since $\widetilde{M}$ is a global product, then h defines a metric on N. Moreover, the group $\mathbb{Z}$ is a group of isometries of $(\widetilde{M}, \widetilde{g})$ and hence, integer powers of f are isometries of (N, h). Thus f is an isometry of (N, h). Therefore, we have proved the following.

Theorem 4.8. (Farran [Far81]). *Let $f : N \longrightarrow N$ be a diffeomorphism, $\widetilde{M} = \mathbb{R} \times N$ and $M = \widetilde{M}/\mathbb{Z}$ as above. Then M admits a semi–Riemannian metric such that $\mathcal{F}$ and $\mathcal{F}'$ are parallel, non–degenerate and mutually orthogonal, if and only if N admits a semi–Riemannian metric with respect to which f is an isometry.*

4.5 Parallel Totally–Null Foliations

Let $\mathcal{F}$ be a totally–null r–foliation on an m–dimensional proper semi–Riemannian manifold (M, g). Thus using the notations introduced in Section 4.3, $\mathcal{F}$ is of type $(r, 0)$, $r > 0$. If $\mathcal{D}$ is the tangent distribution to $\mathcal{F}$, then $\mathcal{D} = \mathcal{N} = \mathcal{D} \cap \mathcal{D}^{\perp}$, and hence $\mathcal{D} \subset \mathcal{D}^{\perp}$ and $\mathcal{D}^{+} = \mathcal{D}^{\perp}$. Therefore $\mathcal{D}^{\perp}$ can be thought of as a partially–null $(r+u)$–distribution, provided $u > 0$. Thus, in this section we have $m = 2r + u$ where both r and u are positive integers. Now, we take $m = 2r + u$ in Theorem 3.2 and obtain the following.

Theorem 5.1. *Let $\mathcal{D}$ be a parallel totally–null distribution of type $(r, 0)$ on a $(2r+u)$–dimensional proper semi–Riemannian manifold (M, g). Then M admits a foliated atlas $\mathcal{A}$ in which the transformations of coordinates are given by*

$$\widetilde{x} = \widetilde{x}(x, z, t), \; \widetilde{z} = \widetilde{z}(z, t), \; \widetilde{t} = \widetilde{t}(t). \tag{5.1}$$

Moreover, the distributions $\mathcal{D}, \mathcal{D}^{\perp}$ are locally spanned by

$$\begin{aligned} &(a) \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r} \right\}, \\ &(b) \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^u} \right\}, \end{aligned} \tag{5.2}$$

respectively.

Hence, at an arbitrary point of M there exists a foliated chart $(\mathcal{U}, \varphi)$ with local coordinates $(x^1, ..., x^r, z^1, ..., z^u, t^1, ..., t^r)$ such that the plaques of $\mathcal{F}^\perp$ and $\mathcal{F}$ are given by the equations $t^i = b^i$ and $t^i = b^i$, $z^\alpha = c^\alpha$, respectively, where $i \in \{1, ..., r\}$, $\alpha \in \{1, ..., u\}$.

As in the case of parallel non–degenerate foliations, the first step in studying the geometry of (M, g) endowed with the totally–null foliation $\mathcal{F}$, is to find a foliated atlas for which the metric has a certain canonical form (see (4.1) in the non–degenerate case). In the present case, the canonical form of g was found by Walker [Wal50a].

Theorem 5.2. (Walker [Wal50a]). *Let (M, g) be a $(2r+u)$–dimensional proper semi–Riemannian manifold, and $\mathcal{F}$ an r–foliation on M. Then $\mathcal{F}$ is a parallel totally–null foliation if and only if there is a foliated atlas $\mathcal{A}$ on M satisfying* (5.1) *and* (5.2) *with respect to which the matrix of g takes the canonical form*

$$\begin{bmatrix} 0 & 0 & I_r \\ 0 & A(z,t) & H(z,t) \\ I_r & H^T(z,t) & B(x,z,t) \end{bmatrix}, \tag{5.3}$$

where the non–zero submatrices satisfy the following conditions:

(i) *I_r is the $r \times r$ identity matrix. A is a non–singular symmetric $u \times u$ matrix and B is a symmetric $r \times r$ matrix. H is of size $u \times r$ and H^T is the transpose of H.*
(ii) *A and H (and therefore H^T) are independent of $(x^1, ..., x^r)$.*

Proof. First, suppose that $\mathcal{F}$ is a parallel totally–null r–foliation on (M, g). Then by Theorem 5.1 there exists an atlas $\mathcal{A}$ on M satisfying (5.1) and (5.2). Let $(\mathcal{U}, \varphi)$ be a foliated chart from $\mathcal{A}$ with local coordinates (x^i, z^α, t^i), where $i \in \{1, ..., r\}$ and $\alpha \in \{1, ..., u\}$. Since $\mathcal{F}$ is totally–null we have

$$(a)\ g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = 0, \quad (b)\ g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial z^\alpha}\right) = g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial x^i}\right) = 0, \tag{5.4}$$

which justify the existence of the zero submatrices in (5.3). Now, we consider the vector fields $\{\xi_i\}$, $i \in \{1, ..., r\}$ defined on $\mathcal{U}$ by

$$g(\xi_i, X) = dt^i(X), \quad \forall\, X \in \Gamma(TM_{|\mathcal{U}}). \tag{5.5}$$

Then it follows that $\{\xi_i\}$ are orthogonal to any $X \in \Gamma(\mathcal{D}^\perp)$ and hence they lie in $\Gamma(\mathcal{D})$. Moreover, they are linearly independent since $\{dt^i\}$ are so. Next, let $\left\{\dfrac{\partial}{\partial x^a}\right\}$, $a \in \{1, ..., 2r+u\}$ be the local frames field, where we have

$$\frac{\partial}{\partial x^i} \in \Gamma(\mathcal{D}), \ \frac{\partial}{\partial x^{r+\alpha}} = \frac{\partial}{\partial z^\alpha}, \ \frac{\partial}{\partial x^{r+u+i}} = \frac{\partial}{\partial t^i}, \quad \begin{array}{l} i \in \{1, ..., r\}, \\ \alpha \in \{1, ..., u\}. \end{array} \tag{5.6}$$

Now, we put

$$\text{(a) } g_{ab} = g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right), \quad \text{(b) } \xi_i = \xi_i^a \frac{\partial}{\partial x^a}, \tag{5.7}$$

and from (5.5) we deduce that

$$\text{(a) } g_{ab}\xi_i^b = \delta_a^{i^*}, \quad \text{(b) } \xi_i^a = g^{ab}\delta_b^{i^*}, \quad i^* = r+u+i. \tag{5.8}$$

By using (5.8a) and taking into account that $\xi_i \in \Gamma(\mathcal{D})$ for all $i \in \{1, ..., r\}$, we obtain

$$\xi_i^a \delta_a^{j^*} = 0, \quad \forall j \in \{1, ..., r\}. \tag{5.9}$$

Also, since $\mathcal{D}$ is parallel with respect to the Levi–Civita connection $\widetilde{\nabla}$ on (M, g), there exist some functions ${A_i}^k{}_b$ on $\mathcal{U}$ such that

$$\xi^a_{i|b} = {A_i}^k{}_b \xi_k^a, \tag{5.10}$$

where $|$ represents the covariant derivative with respect to $\widetilde{\nabla}$. Now, by direct calculations using (5.8), (5.9) and (5.10) we infer that

$$\begin{aligned} \xi_i^a \xi^b_{j|a} &= g^{ac}\delta_c^{i^*}(g^{bd}\delta_d^{j^*})_{|a} = g^{ac}\delta_c^{i^*} g^{bd}(\delta^{j^*}_{a|d}) \\ &= \xi^c_{j|d}\delta_c^{i^*} g^{bd} = {A_j}^k{}_d \xi_k^c \delta_c^{i^*} g^{bd} = 0. \end{aligned}$$

Thus we obtain

$$[\xi_i, \xi_j] = (\xi_i^a \xi^b_{j|a} - \xi_j^a \xi^b_{i|a}) \frac{\partial}{\partial x^b} = 0.$$

Then by Lemma 2.1.6 there exists a local chart $(\overline{\mathcal{U}}, \overline{\varphi})$ on M with coordinates $(\bar{x}^a)$ such that $\xi_i = \dfrac{\partial}{\partial \bar{x}^i}\cdot$ Then we choose the coordinates $(\bar{x}^i, z^\alpha, t^i)$ on $\overline{\mathcal{U}} \cap \mathcal{U}$, and taking into account that (5.8a) is invariant with respect to the transformations of coordinates, we obtain

$$\bar{\xi}_i^b = \delta_i^b \ \text{ and } \ \bar{g}_{j^*i} = \delta_{j^*}^{i^*}.$$

Thus there exists an atlas $\mathcal{A}$ satisfying (5.1) and (5.2) and with respect to which (we omit the bar)

$$g_{ij^*} = \delta_{i^*j^*}. \tag{5.11}$$

This proves the existence of the matrix I_r in (5.3).

Next, we show that A and H are independent of $(x^1, ..., x^r)$. First, taking into account that both $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel with respect to $\widetilde{\nabla}$ we have

$$\begin{aligned}&(a)\ g\left(\widetilde{\nabla}_{\frac{\partial}{\partial z^\alpha}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial z^\beta}\right) = 0, \quad (b)\ g\left(\widetilde{\nabla}_{\frac{\partial}{\partial t^j}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial z^\alpha}\right) = 0,\\ &\qquad\qquad (c)\ g\left(\widetilde{\nabla}_{\frac{\partial}{\partial t^j}}\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial x^i}\right) = 0.\end{aligned} \tag{5.12}$$

Then, by direct calculations using (5.12) and (5.11), and taking into account that $\widetilde{\nabla}$ is a torsion–free metric connection (see (1.5.8) and (1.5.9)) we obtain

$$\begin{aligned}\frac{\partial}{\partial x^i} g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) &= g\left(\widetilde{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) + g\left(\frac{\partial}{\partial z^\alpha}, \widetilde{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial z^\beta}\right)\\ &= g\left(\widetilde{\nabla}_{\frac{\partial}{\partial z^\alpha}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial z^\beta}\right) + g\left(\frac{\partial}{\partial z^\alpha}, \widetilde{\nabla}_{\frac{\partial}{\partial z^\beta}}\frac{\partial}{\partial x^i}\right) = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x^i} g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial t^j}\right) &= g\left(\widetilde{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial t^j}\right) + g\left(\frac{\partial}{\partial z^\alpha}, \widetilde{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial t^j}\right)\\ &= g\left(\widetilde{\nabla}_{\frac{\partial}{\partial z^\alpha}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t^j}\right) + g\left(\frac{\partial}{\partial z^\alpha}, \widetilde{\nabla}_{\frac{\partial}{\partial t^j}}\frac{\partial}{\partial x^i}\right)\\ &= -g\left(\frac{\partial}{\partial x^i}, \widetilde{\nabla}_{\frac{\partial}{\partial z^\alpha}}\frac{\partial}{\partial t^j}\right) = -g\left(\frac{\partial}{\partial x^i}, \widetilde{\nabla}_{\frac{\partial}{\partial t^j}}\frac{\partial}{\partial z^\alpha}\right) = 0.\end{aligned}$$

Thus the matrices A and H (and therefore H^T) are independent of $(x^1, ..., x^r)$. This completes the proof of the assertions (i) and (ii).

Conversely, suppose $\mathcal{F}$ is an r–foliation on (M, g) and there exists a foliated atlas $\mathcal{A}$ satisfying (5.1) and (5.2) with respect to which g has the canonical form (5.3). Then the zero matrix from the corner of the matrix in (5.3) indicates that $\mathcal{F}$ is a totally–null foliation. Next, we put

$$\widetilde{\nabla}_{\frac{\partial}{\partial x^a}}\frac{\partial}{\partial x^i} = A_i{}^k{}_a\,\frac{\partial}{\partial x^k} + B_i{}^\alpha{}_a\,\frac{\partial}{\partial z^\alpha} + C_i{}^k{}_a\,\frac{\partial}{\partial t^k}\,\cdot \tag{5.13}$$

From (5.3) we deduce that

$$g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial t^k}\right) = \delta_{jk}, \tag{5.14}$$

and thus (5.13) implies

$$C_i{}^j{}_a = g\left(\widetilde{\nabla}_{\frac{\partial}{\partial x^a}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\cdot$$

Also, from (5.3) and condition (ii) we obtain

$$\text{(a)}\ \frac{\partial}{\partial x^i}\, g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^a}\right) = 0, \quad \text{(b)}\ \frac{\partial}{\partial x^i}\, g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial x^a}\right) = 0,$$
$$\text{(c)}\ \frac{\partial}{\partial z^\alpha}\, g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^i}\right) = 0. \tag{5.15}$$

Now, by using (1.5.10) for $\widetilde{\nabla}$, and (5.15a) we infer that

$$2g\left(\widetilde{\nabla}_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^a}\, g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + \frac{\partial}{\partial x^i}\, g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^a}\right)$$
$$- \frac{\partial}{\partial x^j}\, g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^i}\right) = 0.$$

Hence $C_i{}^j{}_a = 0$ and thus (5.13) becomes

$$\widetilde{\nabla}_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^i} = A_i{}^k{}_a \frac{\partial}{\partial x^k} + B_i{}^\alpha{}_a \frac{\partial}{\partial z^\alpha}\,. \tag{5.16}$$

Now, from (5.16) we obtain

$$g\left(\widetilde{\nabla}_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial z^\beta}\right) = B_i{}^\alpha{}_a\, A_{\alpha\beta}, \text{ where } A_{\alpha\beta} = g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right).$$

By similar calculations as above, using (1.5.10), (5.15b) and (5.15c) we deduce that

$$g\left(\widetilde{\nabla}_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial z^\beta}\right) = 0.$$

Since the matrix A from (5.3) is non–singular and $A_{\alpha\beta}$ are its entries, we infer that $B_i{}^\alpha{}_a = 0$. Hence (5.16) becomes

$$\widetilde{\nabla}_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^i} = A_i{}^k{}_a \frac{\partial}{\partial x^k}\,.$$

Thus $\mathcal{F}$ is a parallel totally–null foliation. This completes the proof of the theorem. ∎

The atlas $\mathcal{A}$ given by Theorem 5.2 will be called a **Walker atlas**. Now, since in a Walker atlas, the change of coordinates preserves the canonical form (5.3) of the metric g, we expect that these coordinate transformations take a special form. To express this explicitly we start with (5.1) from which we deduce that

$$\text{(a)}\ \frac{\partial}{\partial x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\partial}{\partial \widetilde{x}^j}, \quad \text{(b)}\ \frac{\partial}{\partial z^\alpha} = \frac{\partial \widetilde{x}^j}{\partial z^\alpha} \frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{z}^\beta}{\partial z^\alpha} \frac{\partial}{\partial \widetilde{z}^\beta},$$
$$\text{(c)}\ \frac{\partial}{\partial t^i} = \frac{\partial \widetilde{x}^j}{\partial t^i} \frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{z}^\beta}{\partial t^i} \frac{\partial}{\partial \widetilde{z}^\beta} + \frac{\partial \widetilde{t}^k}{\partial t^i} \frac{\partial}{\partial \widetilde{t}^k}\,. \tag{5.17}$$

Then, by direct calculations, taking into account that (5.14) is true for any local chart of a Walker atlas, we obtain

$$\delta_{ij} = \frac{\partial \widetilde{x}^h}{\partial x^i}\,\frac{\partial \widetilde{t}^k}{\partial t^j}\,\delta_{hk},$$

which implies

$$\frac{\partial \widetilde{x}^i}{\partial x^j} = L^i_j(t), \quad \text{where } L^i_j(t) = \frac{\partial t^j}{\partial \widetilde{t}^i}\cdot \tag{5.18}$$

Thus the coordinate transformations in a Walker atlas are given by

$$\begin{aligned} \widetilde{x}^i &= L^i_j(t)x^j + S^i(z,t), \\ \widetilde{z}^\alpha &= \widetilde{z}^\alpha(z,t), \\ \widetilde{t}^i &= \widetilde{t}^i(t). \end{aligned} \tag{5.19}$$

Taking into account that the canonical form (5.3) is preserved with respect to the coordinate transformations (5.19), from Theorem 5.2 we deduce the following.

Theorem 5.3. *Let M be a $(2r+u)$–dimensional manifold that admits an atlas in which the change of coordinates is given by* (5.19)*. If $\mathcal{F}$ is the r–foliation whose tangent distribution is locally represented by $\left\{\frac{\partial}{\partial x^i}\right\}$, then there exists on M a proper semi–Riemannian metric g such that $\mathcal{F}$ is totally–null and parallel with respect to the Levi–Civita connection on (M,g).*

To state the next result on the leaves of $\mathcal{F}$ we introduce a special class of manifolds. Let N be an r–dimensional manifold and ∇ be a linear connection on N. Then ∇ is locally represented by r^3 smooth functions ${\mathit{\Gamma}_i}^k{}_j$ satisfying (Kobayashi–Nomizu [KN63], p. 141)

$$\widetilde{\mathit{\Gamma}}_\ell{}^h{}_p\,\frac{\partial \widetilde{x}^\ell}{\partial x^i}\,\frac{\partial \widetilde{x}^p}{\partial x^j} - {\mathit{\Gamma}_i}^k{}_j\,\frac{\partial \widetilde{x}^h}{\partial x^k} = \frac{\partial^2 \widetilde{x}^h}{\partial x^i \partial x^j}\,, \tag{5.20}$$

with respect to a coordinate transformation $\widetilde{x}^i = \widetilde{x}^i(x^j)$. Then the local components ${T_i}^k{}_j$ and ${R_i}^k{}_{j\ell}$ of the torsion tensor field T and the curvature tensor field R with respect to the natural frame field $\left\{\frac{\partial}{\partial x^i}\right\}$ are given by

$${T_i}^k{}_j = {\mathit{\Gamma}_i}^k{}_j - {\mathit{\Gamma}_j}^k{}_i, \tag{5.21}$$

and

$${R_i}^k{}_{jh} = \frac{\partial {\mathit{\Gamma}_i}^k{}_j}{\partial x^h} - \frac{\partial {\mathit{\Gamma}_i}^k{}_h}{\partial x^j} + {\mathit{\Gamma}_i}^s{}_j\,{\mathit{\Gamma}_s}^k{}_h - {\mathit{\Gamma}_i}^s{}_h\,{\mathit{\Gamma}_s}^k{}_j. \tag{5.22}$$

When both T and R vanish identically on N we say that N is a **locally affine manifold** and (N,∇) is a **locally affine structure**. To justify this name we consider the system of partial differential equations

$$\frac{\partial^2 \widetilde{x}^h}{\partial x^i \partial x^j} + \Gamma_i{}^k{}_j \frac{\partial \widetilde{x}^h}{\partial x^k} = 0,$$

which has solutions $(\widetilde{x}^h)$, $h \in \{1, ..., r\}$, provided $T = 0$ and $R = 0$ on N. Then by (5.20) we deduce that $\widetilde{\Gamma}_\ell{}^h{}_p = 0$ for all $h, \ell, p \in \{1, ..., r\}$. Thus there exists an atlas on N with respect to which all the connection coefficients vanish, and hence (5.20) implies

$$\frac{\partial^2 \widetilde{x}^h}{\partial x^i \partial x^j} = 0.$$

Thus the coordinate transformations on N must be affine transformations

$$\widetilde{x}^i = a^i_j x^j + b^i, \tag{5.23}$$

where a^i_j and b^i are constant. Conversely, if on N there exists an atlas satisfying (5.23), then by (5.20), $\Gamma_i{}^k{}_j = 0$ with respect to any local chart, determine a globally defined linear connection with $T = 0$ and $R = 0$. An atlas on N with coordinate transformations given by (5.23) is called an **affine atlas**. Then based on the above discussion we can state the following.

Theorem 5.4. (Auslander–Marcus [AM55]). *A smooth manifold N is locally affine if and only if there exists on N an affine atlas.*

The most familiar example of a locally affine manifold is $(\mathbb{E}^n, \widetilde{\nabla})$, where $\mathbb{E}^n$ is the Euclidean n-space and $\widetilde{\nabla}$ is the standard Euclidean connection on $\mathbb{E}^n$.

Next, in order to state a result on the global structure of a locally affine manifold, we give the following definitions. Let ∇ and ∇' be two linear connections on N and N' and $f : N \longrightarrow N'$ be a smooth map. Then we say that f is a **connection preserving map** if it satisfies

$$f_*(\nabla_X Y) = \nabla'_{f_* X} f_* Y, \quad \forall X, Y \in \Gamma(TM).$$

When f is both a diffeomorphism and a connection preserving map we say that it is an **affine equivalence** of (N, ∇) and (N', ∇'). Now, we can state the following.

Theorem 5.5. (Auslander–Marcus [AM55], Wolf [Wol67]). *Every complete locally affine n–dimensional manifold (N, ∇) is affinely equivalent to $(\mathbb{E}^n, \widetilde{\nabla})/G$, where G is some properly discontinuous group of automorphisms of $(\mathbb{E}^n, \widetilde{\nabla})$.*

Now, suppose $\mathcal{F}$ is an n–foliation on an $(n + p)$–dimensional manifold M. Denote by $C(M, \mathcal{F})$ the class of torsion–free linear connections on M with respect to which $\mathcal{F}$ is parallel. Proposition 1.5 guarantees that $C(M, \mathcal{F})$ is non–empty. Let N be a leaf of $\mathcal{F}$ and $\widetilde{\nabla} \in C(M, \mathcal{F})$. Then $\widetilde{\nabla}$ induces a torsion–free linear connection ∇ on N given by

$$\nabla_X Y = \widetilde{\nabla}_X Y, \quad \forall X, Y \in \Gamma(TN).$$

The foliation $\mathcal{F}$ is called **locally affine** if there exists $\widetilde{\nabla} \in C(M, \mathcal{F})$ which induces on each leaf N of $\mathcal{F}$ a locally affine structure (N, ∇). Next, we denote by (x^i, x^α), $i \in \{1, ..., n\}$, $\alpha \in \{n+1, ..., n+p\}$ the local coordinates on M with respect to the leaf atlas on $(M, \mathcal{F})$ (see Section 2.1). Then a local characterization of locally affine foliations can be stated as follows.

Theorem 5.6. (Furness [Fur72], p. 35). *The foliation $\mathcal{F}$ is locally affine on M if and only if there exists a leaf atlas on $(M, \mathcal{F})$ with coordinate transformations given by*

$$\begin{aligned} \widetilde{x}^i &= A^i_j(x^\alpha)x^j + B^i(x^\alpha), \quad && i, j \in \{1, ..., n\}, \\ \widetilde{x}^\alpha &= C^\alpha(x^\beta), && \alpha, \beta \in \{n+1, ..., n+p\}. \end{aligned} \tag{5.24}$$

This theorem is a generalization of Theorem 5.4 and we omit its proof here. Comparing (5.24) and (2.1.21) we may state the following.

Corollary 5.7. *The vertical foliation on the total space of a vector bundle is locally affine.*

Another large class of locally affine foliations is provided by the next theorem.

Theorem 5.8. (Furness [Fur72], p. 42). *Any 1–foliation on a paracompact manifold is locally affine.*

Now, let $\mathcal{F}$ be a parallel totally–null r–foliation on a semi–Riemannian manifold (M, g). Then comparing the coordinate transformations (5.19) in a Walker atlas with (5.24) we can state the following.

Theorem 5.9. *Any parallel totally–null foliation on a semi–Riemannian manifold is locally affine.*

The above theorem is a particular case of a general result obtained by Robertson–Furness [RF74] (see Theorem 7.2).

We have no universal model for manifolds admitting a parallel totally–null foliation. It is important, therefore, to look for general constructions of such foliations. One such construction is the following.

Let M be an $(n+p)$–dimensional manifold and $\mathcal{F}$ be an n–foliation on M. Then we consider the tangent distribution $\mathcal{D}$ to $\mathcal{F}$ and define a vector subbundle $\mathcal{D}^*$ of the cotangent bundle T^*M as follows. For each $x \in M$ the fiber $\mathcal{D}^*_x$ of $\mathcal{D}^*$ consists of all linear mappings $\omega : T_xM \longrightarrow \mathbb{R}$ such that $\omega(X) = 0$ for all $X \in \mathcal{D}_x$. We call $\mathcal{D}^*$ the **conormal bundle** of $\mathcal{F}$ on M. It is easy to see that $\mathcal{D}^*$ is bundle isomorphic to $TM/\mathcal{D}$ and therefore its fiber dimension is p. Thus the total space of the vector bundle $\pi^* : \mathcal{D}^* \longrightarrow M$ is an $(n+2p)$–dimensional manifold and the fibers $\mathcal{D}^*_x$, $x \in M$, are the leaves of a p–foliation $\mathcal{G}$ on $\mathcal{D}^*$.

Theorem 5.10. *There exists a semi–Riemannian metric on $\mathcal{D}^*$ with respect to which the foliation $\mathcal{G}$ is totally–null and parallel.*

Proof. Let $\mathcal{A}$ be a leaf atlas for the n–foliation $\mathcal{F}$ on M with coordinate transformations (see (2.1.5))

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^j, x^\beta),\ i, j \in \{1, ..., n\}, \\ &\widetilde{x}^\alpha = \widetilde{x}^\alpha(x^\beta), \quad \alpha, \beta \in \{n+1, ...n+p\}. \end{aligned} \tag{5.25}$$

Then $\mathcal{A}$ induces an atlas $\mathcal{A}^*$ on the $(n+2p)$–dimensional manifold $\mathcal{D}^*$ as follows. Locally, $\omega \in \Gamma(\mathcal{D}^*)$ is written $\omega = y_\beta dx^\beta$. Then the coordinates on $\mathcal{D}^*$ are taken (x^i, x^α, y_β), $i \in \{1, ..., n\}$, $\alpha, \beta \in \{n+1, ..., n+p\}$. By using (5.25) and taking into account that ω is an 1–form on M we deduce that the coordinate transformations of $\mathcal{A}^*$ have the following form

$$\begin{aligned} &\widetilde{y}_\alpha = L^\beta_\alpha(x^\gamma) y_\beta,\ L^\beta_\alpha(x^\gamma) = \frac{\partial x^\beta}{\partial \widetilde{x}^\alpha}, \\ &\widetilde{x}^i = \widetilde{x}^i(x^j, x^\beta), \\ &\widetilde{x}^\alpha = \widetilde{x}^\alpha(x^\beta). \end{aligned} \tag{5.26}$$

Comparing (5.26) with (5.19) we deduce that $\mathcal{A}^*$ is a Walker atlas on $\mathcal{D}^*$, and the assertion follows from Theorem 5.3. ■

Finally, we note that locally, the leaves of the foliation $\mathcal{G}$ are given by $x^i = \text{const.}$, $x^\alpha = \text{const.}$, while the leaves of the orthogonal foliation $\mathcal{G}^\perp$ are given by $x^\alpha = \text{const.}$ For convenience, we refer to a totally–null foliation constructed in the above fashion as a **totally–null conormal bundle foliation** to $\mathcal{F}$.

Another source of examples for parallel totally–null r–foliations, for $r = 1$ this time, is the technique of suspensions discussed in Section 4.4. So, let N be an $(m-1)$–dimensional manifold and $f : N \longrightarrow N$ be a diffeomorphism. Suppose $M = \widetilde{M}/\mathbb{Z}$ is the foliated manifold obtained by suspension of f, and let $\mathcal{F}$ be the 1–foliation on M. As in the non–degenerate case, we look for necessary and sufficient conditions for the existence of a semi–Riemannian metric on M such that $\mathcal{F}$ is totally–null and parallel.

First, we need the following definitions. Let (N, h) be a complete Riemannian manifold and $f : N \longrightarrow N$ a diffeomorphism. Then f is said to be **expanding** if there exist real numbers $c > 0$ and $\lambda > 1$ such that

$$\|Tf^n(v)\| \geq c\lambda^n \|v\|, \tag{5.27}$$

for all $v \in TM$ and all positive integers n, where Tf^n is the differential of f^n and $\|\cdot\|$ is the norm on TM given by h. The diffeomorphism f is **contracting** if there exist real numbers $c > 0$ and $0 < \lambda < 1$ such that

$$\|Tf^n(v)\| \leq c\lambda^n \|v\|. \tag{5.28}$$

Obviously, if f is expanding then f^{-1} is contracting and viceversa. It is well known (see Nitecki [Nit71] and Shub [Shu69]) that if f is expanding or contracting then it has a unique fixed point, and when N is compact, the property of being expanding or contracting is independent of the choice of the metric. Now, we are in a position to prove the following.

Theorem 5.11. (Farran [Far81]). *Let $f : N \longrightarrow N$ be a diffeomorphism and $M = \widetilde{M}/\mathbb{Z}$ be the manifold obtained by the suspension of f, and take $\mathcal{F}$ as the induced 1–foliation on M. If M admits a proper semi–Riemannian metric for which $\mathcal{F}$ is totally–null and parallel, then f cannot be either expanding or contracting.*

Proof. Recall from Section 4.4 that $\widetilde{M} = \mathbb{R}\times N$ and for each $i \in \mathbb{Z}$ we have a diffeomorphism $\Phi_i : \widetilde{M} \longrightarrow \widetilde{M}$ given by

$$\Phi_i(t, x) = (t + i, f^i(x)).$$

On the $(m-1)$–dimensional manifold N we consider an atlas $\mathcal{A}=\{(\mathcal{W}_\alpha, \psi_\alpha)\}_{\alpha\in A}$ and take the open sets of $\widetilde{M}$:

$$\mathcal{U}_\alpha = (0, 1)\times\mathcal{W}_\alpha \quad \text{and} \quad \mathcal{V}_\alpha = \left(-\frac{1}{2}, \frac{1}{2}\right)\times\mathcal{W}_\alpha.$$

Then we define $\varphi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathbb{R}^m$ and $\eta_\alpha : \mathcal{V}_\alpha \longrightarrow \mathbb{R}^m$, by

$$\varphi_\alpha(t, x) = (t, \psi_\alpha(x)) \quad \text{and} \quad \eta_\alpha(s, y) = (s, \psi_\alpha(y)).$$

Since the natural projection $p : \widetilde{M} \longrightarrow M$ is injective on each of $\mathcal{U}_\alpha$ and $\mathcal{V}_\alpha$, then $\overline{\mathcal{U}}_\alpha = p(\mathcal{U}_a)$ and $\overline{\mathcal{V}}_\alpha = p(\mathcal{V}_\alpha)$ are open sets of M on which we can define the following:

$$\overline{\varphi}_\alpha : \overline{\mathcal{U}}_\alpha \longrightarrow \mathbb{R}^m, \overline{\varphi}_\alpha = \varphi_\alpha \circ p^{-1}, \quad \text{and}$$
$$\overline{\eta}_\alpha : \overline{\mathcal{V}}_\alpha \longrightarrow \mathbb{R}^m, \ \overline{\eta}_\alpha = \eta_\alpha \circ p^{-1}.$$

Thus, $(\overline{\mathcal{U}}_\alpha, \overline{\varphi}_\alpha)$ and $(\overline{\mathcal{V}}_\alpha, \overline{\eta}_\alpha)$ are two local charts in the atlas on M induced by the atlas $\mathcal{A}$ on N. Now, if $f : N \longrightarrow N$ has no fixed points, then it cannot be expanding or contracting, and we are done. So let us assume that f has at least one fixed point, say x. Let $(\mathcal{W}_\alpha, \psi_\alpha)$ be a chart in $\mathcal{A}$ about x, and $(\overline{\mathcal{U}}_\alpha, \overline{\varphi}_\alpha)$, $(\overline{\mathcal{V}}_\alpha, \overline{\eta}_\alpha)$ the corresponding two charts of M as above. Now, $\overline{\mathcal{U}}_\alpha \cap \overline{\mathcal{V}}_\alpha = P \cup Q$ where P and Q are connected components which come under p from $\left(0,\dfrac{1}{2}\right)\times\mathcal{W}_\alpha$ and $\left(\dfrac{1}{2},1\right)\times\mathcal{W}_\alpha$ respectively (see Brickell–Clark [BC70], p. 104). The change of coordinates $\overline{\eta}_\alpha\circ\overline{\varphi}_\alpha^{-1}$ on $\overline{\varphi}_\alpha(P)$ is given by $(t,x)\longrightarrow(t,x)$ and it arises from the identity Φ_0. The change of coordinates $\overline{\eta}_\alpha\circ\overline{\varphi}_\alpha^{-1}$ on $\overline{\varphi}_\alpha(Q)$ is given by $(t,x)\longrightarrow(t-1,f^{-1}(x))$ and it arises from Φ_{-1}. So the change of coordinates on Q is given by

$$\widetilde{t}=t-1,\;\; \widetilde{x}^i=(f^i)^{-1}(x^1,...,x^{m-1}),\quad i\in\{1,...,m-1\}. \tag{5.29}$$

Now, if M admits a semi–Riemannian metric for which $\mathcal{F}$ is totally–null and parallel, then there is an atlas $\mathcal{A}$ on N such that the induced atlas on M is a Walker atlas. Thus, by (5.19) and (5.18) the change of coordinates in that Walker atlas must be of the form

$$\begin{aligned}
&\widetilde{t}=\frac{dx^{m-1}}{d\widetilde{x}^{m-1}}\,t+S(x^1,...,x^{m-1}),\\
&\widetilde{x}^\alpha=\widetilde{x}^\alpha(x^1,...,x^{m-1}),\;\;\alpha\in\{1,...,m-2\},\\
&\widetilde{x}^{m-1}=\widetilde{x}^{m-1}(x^{m-1}).
\end{aligned} \tag{5.30}$$

Comparing (5.29) and (5.30) we conclude that $\widetilde{x}^{m-1}=x^{m-1}+c$ where c is a real constant. Using this, we deduce that $T_xf^{-1}:T_xM\longrightarrow T_xM$ has a matrix of the form

$$\begin{bmatrix} A & b\\ 0 & 1\end{bmatrix},$$

where A is a non singular $(m-1)\times(m-1)$ matrix and $b\in\mathbb{R}$. Therefore, T_xf^{-1} has at least one eigenvalue which is equal to 1. Thus f can not be either expanding or contracting, which completes the proof of the theorem. ∎

Theorem 5.12. (Farran [Far81]). *Let (N,h) be an $(m-1)$–dimensional Riemannian manifold admitting a parallel 1–foliation, and let $f:N\longrightarrow N$ be a diffeomorphism. If f is an isometry of (N,h) then there exists a semi–Riemannian metric on $M=\widetilde{M}/\mathbb{Z}$ such that the 1–foliation $\mathcal{F}$ obtained by suspension of f is parallel and totally–null.*

Proof. Since (N,h) admits a parallel 1–foliation, by using Theorems 4.1 and 5.5 we deduce that there exists an atlas $\mathcal{A}$ on N in which the change of coordinates is given by

$$\begin{aligned}
&\widetilde{x}^i=\widetilde{x}^i(x^1,...,x^{m-2}),\; i\in\{1,...,m-2\}.\\
&\widetilde{x}^{m-1}=x^{m-1}+c,\qquad c \text{ is a real constant.}
\end{aligned}$$

So, $\widetilde{M}=R\times N$ admits an atlas $\mathcal{B}$ in which the transformations of coordinates are given by

$$\begin{aligned}
&\widetilde{t} = t, \\
&\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^{m-2}), \\
&\widetilde{x}^{m-1} = x^{m-1} + c.
\end{aligned} \tag{5.31}$$

Moreover, the Riemannian metric h must be locally represented by the following matrix

$$\begin{bmatrix} A & 0 \\ 0 & b \end{bmatrix},$$

where A is a non–singular $(m-2)\times(m-2)$ matrix whose entries are functions of $(x^1, ..., x^{m-2})$ and b is a non–zero function of x^{m-1} alone. Now, we take the matrix

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & A & 0 \\ 1 & 0 & b \end{bmatrix},$$

where A and b are as above, which gives a semi–Riemannian metric in every chart of $\mathcal{B}$. But the change of coordinates (5.31) in $\mathcal{B}$ preserves C, and hence we obtain a semi–Riemannian metric ρ on the whole of $\widetilde{M}$. Then, by Theorem 5.2, the foliation on $\widetilde{M}$ locally given by $x^a = c^a$, $a \in \{1, ..., m-1\}$ is parallel and totally–null with respect to ρ. Since f is an isometry of (N, h), then an argument similar to that of the proof of Theorem 4.8 shows that ρ projects to a semi–Riemannian metric g on M. Clearly, $\mathcal{F}$ is parallel and totally null with respect to g. ∎

4.6 Parallel Totally–Null r–Foliations on $2r$–Dimensional Semi–Riemannian Manifolds

Let (M, g) be a $2r$–dimensional proper semi–Riemannian manifold, and $\mathcal{D}$ be a totally–null r–distribution on M, that is, we have

$$g(X, Y) = 0, \quad \forall\, X, Y \in \Gamma(\mathcal{D}). \tag{6.1}$$

In the first part of this section we will construct a complementary totally–null r–distribution $\overline{\mathcal{D}}$ to $\mathcal{D}$ in TM. Then we use $\overline{\mathcal{D}}$ to study the geometry of parallel totally–null r–foliations on M.

First, we consider a complementary distribution $\mathcal{D}'$ to $\mathcal{D}$ in TM that is locally represented on $\mathcal{U} \subset M$ by the vector fields $\{V_1, ..., V_r\}$. Then suppose that $\mathcal{D}$ is locally represented by $\{\xi_1, ..., \xi_r\}$ and consider the $r\times r$ matrices $C = [C_{ij}]$ and $D = [D_{ij}]$ where we put

$$\text{(a) } C_{ij} = g(V_i, \xi_j) \quad \text{and} \quad \text{(b) } D_{ij} = g(V_i, V_j). \tag{6.2}$$

Thus the matrix of g with respect to the non–holonomic frame field $\{\xi_i, V_i\}$, $i \in \{1, ..., r\}$ has the form

$$[g] = \begin{bmatrix} 0 & C \\ C^T & D \end{bmatrix}, \tag{6.3}$$

which implies that C must be nonsingular. Next, we consider the $r \times r$ matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ given by

$$\text{(a) } A = -\frac{1}{2} C^{-1} D (C^{-1})^T \quad \text{and} \quad \text{(b) } B = C^{-1}. \tag{6.4}$$

Then we construct the vector fields

$$\eta_i = \sum_{j=1}^{r} \{A_{ij} \xi_j + B_{ij} V_j\}, \quad i \in \{1, ..., r\}. \tag{6.5}$$

By direct calculations using (6.2) and (6.5) we obtain

$$g(\eta_i, \xi_k) = \sum_{j=1}^{r} B_{ij} C_{jk},$$

and

$$g(\eta_i, \eta_j) = 2A_{ij} + \sum_{k,h=1}^{r} \{B_{ik} D_{kh} B_{jh}\},$$

since A is a symmetric matrix. Hence, by (6.4) we deduce that

$$\text{(a) } g(\eta_i, \xi_k) = \delta_{ik} \quad \text{and} \quad \text{(b) } g(\eta_i, \eta_j) = 0. \tag{6.6}$$

Now, we are in a position to prove the following.

Theorem 6.1. *Let $\mathcal{D}$ be a totally–null r–distribution on a $2r$–dimensional semi–Riemannian manifold (M, g). Then there exists a totally–null r–distribution $\overline{\mathcal{D}}$ complementary to $\mathcal{D}$ in TM and locally represented by the vector fields $\{\eta_i\}$, $i \in \{1, ..., r\}$, given by* (6.5).

Proof. First, by using (6.6a), it is easy to see that $\{\eta_i\}$ are linearly independent on $\mathcal{U} \subset M$. Then we consider another coordinate neighbourhood $\widetilde{\mathcal{U}} \subset M$ such that $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$. The corresponding objects on $\widetilde{\mathcal{U}}$ to the ones defined above on $\mathcal{U}$ will have a tilde. To simplify the calculations we put $[\eta] = [\eta_1, ..., \eta_r]^T$, $[\xi] = [\xi_1, ..., \xi_r]^T$ and $[V] = [V_1, ..., V_r]^T$ on $\mathcal{U}$ and keep the same notation on $\widetilde{\mathcal{U}}$. Since $\mathcal{D}$ and $\mathcal{D}'$ are distributions on M, on $\mathcal{U} \cap \widetilde{\mathcal{U}}$ we have

$$\text{(a) } [\widetilde{\xi}] = E[\xi] \quad \text{and} \quad \text{(b) } [\widetilde{V}] = F[V], \tag{6.7}$$

where E and F are non–singular matrices. Then, by using (6.2) and (6.7), we obtain

$$\text{(a) } \widetilde{C} = FCE^T \quad \text{and} \quad \text{(b) } \widetilde{D} = FDF^T. \tag{6.8}$$

Now, by using (6.4), (6.5), (6.7) and (6.8), we deduce that

$$[\widetilde{\eta}] = -\frac{1}{2}\,\widetilde{C}^{-1}\widetilde{D}(\widetilde{C}^{-1})^T[\widetilde{\xi}] + \widetilde{C}^{-1}[\widetilde{V}]$$
$$= -\frac{1}{2}\,(E^T)^{-1}C^{-1}F^{-1}FDF^T(F^T)^{-1}(C^{-1})^T E^{-1}E[\xi] + (E^T)^{-1}C^{-1}F^{-1}F[V]$$
$$= (E^T)^{-1}\left(-\frac{1}{2}\,C^{-1}D(C^{-1})^T[\xi] + C^{-1}[V]\right) = (E^T)^{-1}[\eta].$$

Hence we have a distribution $\overline{\mathcal{D}}$ on M locally defined by $\{\eta_i\}$, $i \in \{1, ..., r\}$, given by (6.5). According to (6.6b) $\overline{\mathcal{D}}$ is totally–null. Finally, from (6.6a) we deduce that at any point of $\mathcal{U}$ none of the vector fields $\{\eta_i\}$ lies in $\mathcal{D}$. Hence $\mathcal{D}$ and $\overline{\mathcal{D}}$ are complementary totally–null distributions on M. This completes the proof of the theorem. ∎

As we can see from the above proof, the construction of $\overline{\mathcal{D}}$ depends upon the choice of $\mathcal{D}'$ and hence $\overline{\mathcal{D}}$ is not unique. However, as we see below, we may get information on the geometry of $\mathcal{D}$ by using some geometric objects which do not depend on $\overline{\mathcal{D}}$. Indeed, locally we define the functions

$$h_{ijk} = g\left(\widetilde{\nabla}_{\xi_i}\xi_j, \xi_k\right), \tag{6.9}$$

where $\widetilde{\nabla}$ is the Levi–Civita connection on (M, g). Clearly, h_{ijk} are independent of the transversal distribution $\overline{\mathcal{D}}$. Moreover, since g is parallel with respect to $\widetilde{\nabla}$, by using (6.9) and (6.1) we obtain

$$h_{ijk} + h_{ikj} = 0. \tag{6.10}$$

Now, we can state the following.

Theorem 6.2. *Let $\mathcal{D}$ be an integrable totally–null r–distribution on a $2r$–dimensional semi–Riemannian manifold (M, g). Then $\mathcal{D}$ is self–parallel with respect to $\widetilde{\nabla}$.*

Proof. Taking into account that $\mathcal{D}$ is integrable, by using (6.9) and (6.1), we deduce that

$$h_{ijk} = h_{jik}. \tag{6.11}$$

Next, from (6.10), we obtain

$$h_{jki} + h_{jik} = 0 \quad \text{and} \quad h_{kij} + h_{kji} = 0. \tag{6.12}$$

Then, by using (6.10)–(6.12), we obtain $h_{ijk} = 0$. Finally, by using (6.9), we deduce that $\widetilde{\nabla}_{\xi_j}\xi_j \in \Gamma(\mathcal{D})$ for any $i, j \in \{1, ..., r\}$. Hence $\mathcal{D}$ is self–parallel with respect to $\widetilde{\nabla}$. ∎

Next, we consider the foliation $\mathcal{F}_{\mathcal{D}}$ tangent to $\mathcal{D}$ and take a leaf N of $\mathcal{F}_{\mathcal{D}}$. Then, by the above theorem, the restriction of $\widetilde{\nabla}$ to N is a torsion–free linear connection on N. Thus any geodesic of N is a geodesic of (M, g). This enables us to state the following important result on totally–null foliations.

Theorem 6.3. *Any totally–null r–foliation on a $2r$–dimensional semi–Riemannian manifold is totally geodesic.*

Now, we suppose that $\mathcal{F}$ is a parallel totally–null r–foliation on a $2r$–dimensional semi–Riemannian manifold (M, g). If $\mathcal{D}$ is the tangent distribution to $\mathcal{F}$ then $\mathcal{D} = \mathcal{D}^{\perp} = \mathcal{N}$. Then a foliated atlas on (M, g) has the coordinates (x^i, t^i) and $\mathcal{D}$ is locally spanned by $\left\{\dfrac{\partial}{\partial x^i}\right\}$, $i \in \{1, ..., r\}$. Moreover, by a similar proof as of Theorem 5.2, we obtain the following.

Theorem 6.4. *Let (M, g) be a $2r$–dimensional proper semi–Riemannian manifold, and $\mathcal{F}$ an r–foliation on M. Then $\mathcal{F}$ is a parallel totally–null foliation if and only if there is a foliated atlas $\mathcal{A}$ on M with respect to which the matrix of g takes the canonical form*

$$\begin{bmatrix} 0 & I_r \\ I_r & B(x,t) \end{bmatrix}, \tag{6.13}$$

where B is a symmetric $r \times r$ matrix.

We keep for $\mathcal{A}$ the name **Walker atlas** and note that the coordinate transformations in $\mathcal{A}$ are given by (see (5.19))

$$\begin{aligned} &\text{(a) } \widetilde{x}^i = L^i_j(t)x^i + S^i(t),\ L^i_j(t) = \frac{\partial t^j}{\partial \widetilde{t}^i}, \\ &\text{(b) } \widetilde{t}^i = \widetilde{t}^i(t^j). \end{aligned} \tag{6.14}$$

Since the canonical form (6.13) is preserved with respect to the change of coordinates (6.14), by using Theorem 6.4, we deduce the following.

Theorem 6.5. *Let M be a $2r$–dimensional manifold that admits an atlas in which the change of coordinates is given by* (6.14). *If $\mathcal{F}$ is the r–foliation whose tangent distribution is locally represented by $\left\{\dfrac{\partial}{\partial x^i}\right\}$, $i \in \{1, ..., r\}$, then there exists on M a proper semi–Riemannian metric g such that $\mathcal{F}$ is totally–null and parallel with respect to the Levi–Civita connection on (M, g).*

Now, suppose that $\mathcal{F}$ is a parallel and totally–null r–foliation on a $2r$–dimensional semi–Riemannian manifold (M, g). Since the tangent distribution $\mathcal{D}$ to $\mathcal{F}$ is parallel with respect to the Levi–Civita connection $\widetilde{\nabla}$ on (M, g) we put

$$\widetilde{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Gamma_i{}^k{}_j \, \frac{\partial}{\partial x^k} \, . \tag{6.15}$$

Take a leaf N of $\mathcal{F}$ and denote by ∇ the induced connection by $\widetilde{\nabla}$ on N, that is, by (6.15) $\nabla = \widetilde{\nabla}$ on $\Gamma(TN)$. From (6.13), we have $g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial t^h}\right) = \delta_{kh}$, where (x^k, t^h) are coordinates in the Walker atlas on (M, g). Thus (6.15) implies

$$g\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial t^h}\right) = \Gamma_i{}^h{}_j .$$

On the other hand, taking into account (1.5.10) and (6.13), we obtain

$$2g\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial t^h}\right) = \frac{\partial}{\partial x^j}\left(\delta_{ih}\right) + \frac{\partial}{\partial x^i}\left(\delta_{jh}\right) = 0.$$

Hence the leaf N admits a linear connection ∇ whose local coefficients $\Gamma_i{}^h{}_j$ vanish on the domain of each local chart of the Walker atlas. Thus N is a locally affine manifold and therefore the foliation $\mathcal{F}$ is locally affine. Actually, this follows immediately from (6.14) via Theorem 5.6. The above discussion about the induced connection ∇ on N shows a little more than this. Namely, it shows that the curvature tensor field $\widetilde{R}$ of the Levi–Civita connection $\widetilde{\nabla}$ on (M, g) satisfies

$$\widetilde{R}(X, Y)Z = 0, \quad \forall\, X, Y, Z \in \Gamma(\mathcal{D}). \tag{6.16}$$

Moreover, based on this discussion we can state the following.

Theorem 6.6. *Let $\mathcal{D}$ and $\overline{\mathcal{D}}$ be two complementary parallel totally–null r–distributions on a $2r$–dimensional proper semi–Riemannian manifold (M, g). Then we have the assertions:*

(i) *Both foliations $\mathcal{F}$ and $\overline{\mathcal{F}}$ defined by $\mathcal{D}$ and $\overline{\mathcal{D}}$ are locally affine.*
(ii) *M is locally a product of two locally affine manifolds.*
(iii) *The curvature tensor field $\widetilde{R}$ of the Levi–Civita connection on (M, g) satisfies* (6.16) *and*

$$\widetilde{R}(\,\overline{X},\,\overline{Y}\,)\,\overline{Z} = 0, \quad \forall\, \overline{X},\,\overline{Y},\,\overline{Z} \in \Gamma(\overline{\mathcal{D}}). \tag{6.17}$$

We show now that cotangent bundles are natural models for $2r$–dimensional manifolds that admit parallel totally–null r–foliations.

Theorem 6.7. (Patterson–Walker [PW52]). *The cotangent bundle T^*M of a manifold M admits a proper semi–Riemannian metric such that the foliation by fibers of T^*M is parallel and totally–null.*

Proof. Let (t^i, x_i), $i \in \{1, ..., r\}$, be the local coordinates on T^*M, where (t^i) are the local coordinates on M. Then the change of coordinates on T^*M is given by

$$\text{(a) } \widetilde{x}_i = \frac{\partial t^j}{\partial \widetilde{t}^i} x_j, \quad \text{(b) } \widetilde{t}^i = \widetilde{t}^i(t^j). \tag{6.18}$$

Comparing (6.18) with (6.14) via (5.18) we conclude that the natural atlas $\mathcal{A}$ with local coordinates (t^i, x_i) on T^*M is a Walker atlas with respect to the r–foliation $\mathcal{F}$ by fibers of T^*M. Finally, apply Theorem 6.5 and conclude that T^*M admits a proper semi–Riemannian metric with respect to which $\mathcal{F}$ is parallel and totally–null. ∎

Finally, we note that totally–null distributions (foliations) are deeply involved into the geometry of para–Kählerian manifolds. To show this we first present some definitions. Let F be an almost product structure on a $2r$–dimensional manifold M, and g be a semi–Riemannian metric on M such that

$$g(X, FY) + g(Y, FX) = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{6.19}$$

Then we say that (M, F, g) is an **almost para–Hermitian manifold**. If moreover, F is integrable, that is, the Nijenhuis tensor field N of F given by

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + [X, Y], \quad \forall\, X, Y \in \Gamma(TM), \tag{6.20}$$

vanishes identically on M, then (M, F, g) is said to be a **para–Hermitian manifold**. Next, we denote by $\mathcal{D}^+$ and $\mathcal{D}^-$ the eigendistributions of F corresponding to its eigenvalues $(+1)$ and (-1) respectively. As (6.19) is equivalent to

$$g(FX, FY) + g(X, Y) = 0, \quad \forall\, X, Y \in \Gamma(TM), \tag{6.21}$$

we conclude that $\mathcal{D}^+$ and $\mathcal{D}^-$ are totally–null complementary r–distributions on M. Moreover, we have the following.

Proposition 6.8. *The distributions $\mathcal{D}^+$ and $\mathcal{D}^-$ define on a $2r$–dimensional para–Hermitian manifold (M, F, g) two complementary totally geodesic and totally–null r–foliations.*

Proof. Take $X, Y \in \Gamma(\mathcal{D}^+)$ and since $N = 0$ on M, from (6.20) we obtain $F([X, Y]) = [X, Y]$. Hence $[X, Y] \in \Gamma(\mathcal{D}^+)$, that is, $\mathcal{D}^+$ is integrable. Thus $\mathcal{D}^+$ defines a totally–null foliation $\mathcal{F}^+$ on M. Finally, from Theorem 6.3 we deduce that $\mathcal{F}^+$ is a totally geodesic foliation. Similar arguments apply to $\mathcal{D}^-$, defining a totally geodesic and totally–null r–foliation $\mathcal{F}^-$. ∎

Next, a para–Hermitian manifold (M, F, g) is called **para–Kählerian** if F is parallel with respect to the Levi–Civita connection $\widetilde{\nabla}$ on M, that is, we have

$$(\widetilde{\nabla}_X F)Y = 0, \quad \forall\, X, Y \in \Gamma(TM).$$

Examples and several results on the geometry of para–Kählerian manifolds can be found in a survey of Cruceanu, Fortuny and Gadea [CFG96]. Now, by using the above theory of parallel totally–null foliations we can prove the following.

Theorem 6.9. *Let (M, F, g) be a para–Kählerian manifold and $\mathcal{F}^+$ and $\mathcal{F}^-$ be the foliations defined by the eigendistributions $\mathcal{D}^+$ and $\mathcal{D}^-$ of F. Then we have the assertions:*

(i) *$\mathcal{F}^+$ and $\mathcal{F}^-$ are locally affine, parallel and totally–null foliations.*
(ii) *M is locally a product of two locally affine manifolds.*
(iii) *The curvature tensor field $\widetilde{R}$ of $\widetilde{\nabla}$ satisfies*

$$\text{(a) } \widetilde{R}(X,Y)Z = 0 \quad \textit{and} \quad \text{(b) } \widetilde{R}(U,V)W = 0,$$

for any $X, Y, Z \in \Gamma(\mathcal{D}^+)$ and $U, V, W \in \Gamma(\mathcal{D}^-)$.

Proof. By Proposition 6.8 both foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ are totally–null. Then applying Theorem 2.1 we deduce that $\mathcal{F}^+$ and $\mathcal{F}^-$ are parallel with respect to $\widetilde{\nabla}$. Thus M is endowed with two complementary parallel totally–null r–foliations. Hence Theorem 6.6 applies and we obtain all the assertions of the theorem. ∎

An important relation between parallel totally–null r–foliations on $2r$–dimensional semi–Riemannian manifolds and Lagrangian foliations on symplectic manifolds is presented in Section 5.1.

To investigate that relation we need the following result of Robertson and Furness [RF74].

Theorem 6.10. *Let $\mathcal{F}$ be a parallel totally–null r–foliation on a $2r$–dimensional semi–Riemannian manifold (M, g). Then there is a bundle isomorphism $TM \cong \mathcal{D} \oplus \mathcal{D}$, where $\mathcal{D}$ is the tangent distribution to $\mathcal{F}$. Moreover, M admits an almost complex structure J given by*

$$J_x(u, v) = (-v, u), \quad \forall\, x \in M,\ (u, v) \in \mathcal{D}_x \times \mathcal{D}_x.$$

4.7 Parallel Partially–Null Foliations

This section discusses the most general situation of a parallel partially–null foliation $\mathcal{F}$ on an m–dimensional semi–Riemannian manifold (M, g). Using the terminology of Section 4.3, $\mathcal{F}$ is a foliation of type (r, s) with integers $r > 0$ and $s > 0$. As we saw in Theorem 3.1, $\mathcal{F}$ induces three other parallel foliations $\mathcal{F}^{\perp}, \mathcal{F}^{+}$ and $\mathcal{F}_{\mathcal{N}}$ of type (r, u), $(r, s+u)$ and $(r, 0)$ respectively, where r, s, u verify (3.2).

Unlike the non–degenerate situation (see Section 4.4) the geometry of parallel partially–null foliations is very far from being understood. The global structure of semi–Riemannian manifolds admitting such foliations has not been determined yet. Walker [Wal50b] found a canonical form of the semi–Riemannian metric on such manifolds. Robertson and Furness [RF74] used the transformation of coordinates in a Walker atlas to obtain information on the structure of the leaves of the foliation. Under certain additional conditions, some more results concerning the leaves and the manifold were obtained by Furness [Fur72], [Fur74] and Farran [Far79], [Far80].

In what follows we discuss the main ideas and results obtained for parallel partially–null foliations on semi–Riemannian manifolds. First, using the notations from Theorem 3.2 we can state the following characterization of parallel partially–null foliations.

Theorem 7.1. (Walker [Wal50b]). *Let (M, g) be a $(2r+s+u)$–dimensional proper semi–Riemannian manifold and $\mathcal{F}$ be an $(r+s)$–foliation on M. Then $\mathcal{F}$ is a parallel partially–null foliation of type (r, s), if and only if there is a foliated atlas $\mathcal{A}$ on M satisfying* (3.3) *and* (3.4) *with respect to which the matrix of g takes the canonical form*

$$\begin{bmatrix} 0 & 0 & 0 & I_r \\ 0 & A(y,t) & 0 & F(y,t) \\ 0 & 0 & B(z,t) & G(z,t) \\ I_r & F^T(y,t) & G^T(z,t) & C(x,y,z,t) \end{bmatrix}, \tag{7.1}$$

where the non–zero submatrices satisfy the following conditions:

(i) *I_r is the $r\times r$ identity matrix. A and B are non–singular symmetric matrices of sizes $s\times s$ and $u\times u$ respectively. C is a symmetric $r\times r$ matrix. F and G are matrices of sizes $s\times r$ and $u\times r$ respectively with transposes F^T and G^T respectively.*

(ii) *A and F (and therefore F^T) are independent of $(x^1, ..., x^r, z^1, ..., z^u)$. B and G (and therefore G^T) are independent of $(x^1, ..., x^r, y^1, ..., y^s)$.*

The proof of this theorem is a slight extension of the proof of Theorem 5.2, so we omit it here. It is easy to see that Theorem 7.1 is a generalization of both Theorem 5.2 and Theorem 6.4. For the atlas $\mathcal{A}$ we keep the name **Walker atlas**.

In this present section we use the following range of indices: $i, j, k, ... \in \{1, ..., r\}$; $\alpha, \beta, \gamma, ... \in \{1, ..., u\}$; $\lambda, \mu, \nu, ..., \in \{1, ..., s\}$. Also, we keep the notations from Section 4.3 with respect to the tangent distributions to the foliations we study here. Thus $\mathcal{D}, \mathcal{D}^{\perp}$ and $\mathcal{N}$ are tangent distributions to the foliations $\mathcal{F}, \mathcal{F}^{\perp}$ and $\mathcal{F}_{\mathcal{N}}$ respectively.

Now, let $(\mathcal{U}, \varphi)$ and $(\widetilde{\mathcal{U}}, \widetilde{\varphi})$ be two local charts from $\mathcal{A}$ with overlapping domains. If $(x^i, y^\lambda, z^\alpha, t^j)$ and $(\widetilde{x}^i, \widetilde{y}^\lambda, \widetilde{z}^\alpha, \widetilde{t}^j)$ are the local coordinates on $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ respectively, then by using (3.3) we deduce that

$$\begin{aligned}
&(a)\ \frac{\partial}{\partial x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\,\frac{\partial}{\partial \widetilde{x}^j}\,, \\
&(b)\ \frac{\partial}{\partial y^\lambda} = \frac{\partial \widetilde{x}^j}{\partial y^\lambda}\,\frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{y}^\mu}{\partial y^\lambda}\,\frac{\partial}{\partial \widetilde{y}^\mu}\,, \\
&(c)\ \frac{\partial}{\partial z^\alpha} = \frac{\partial \widetilde{x}^j}{\partial z^\alpha}\,\frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{z}^\beta}{\partial z^\alpha}\,\frac{\partial}{\partial \widetilde{z}^\beta}\,, \\
&(d)\ \frac{\partial}{\partial t^i} = \frac{\partial \widetilde{x}^j}{\partial t^i}\,\frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{y}^\mu}{\partial t^i}\,\frac{\partial}{\partial \widetilde{y}^\mu} + \frac{\partial \widetilde{z}^\beta}{\partial t^i}\,\frac{\partial}{\partial \widetilde{z}^\beta} + \frac{\partial \widetilde{t}^j}{\partial t^i}\,\frac{\partial}{\partial \widetilde{t}^j}\,,
\end{aligned} \tag{7.2}$$

on $\mathcal{U} \cap \widetilde{\mathcal{U}}$. By (7.1) we obtain

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t^j}\right) = \delta_{ij} \quad \text{and} \quad (b)\ g\left(\frac{\partial}{\partial \widetilde{x}^h}, \frac{\partial}{\partial \widetilde{t}^k}\right) = \delta_{hk}. \tag{7.3}$$

By using (7.2a) and (7.2d) into (7.3a) and taking into account that $\mathcal{N}$ is orthogonal to both $\mathcal{D}$ and $\mathcal{D}^\perp$, we infer that

$$\delta_{ij} = \frac{\partial \widetilde{x}^h}{\partial x^i}\,\frac{\partial \widetilde{t}^k}{\partial t^j}\,\delta_{hk}. \tag{7.4}$$

From (7.4) it follows that

$$(a)\ \frac{\partial \widetilde{x}^i}{\partial x^j} = L^i_j(t), \quad \text{where (b)}\ L^i_j(t) = \frac{\partial t^j}{\partial \widetilde{t}^i}\,\cdot \tag{7.5}$$

Thus (3.3) and (7.5) imply the following coordinate transformations in $\mathcal{A}$:

$$\begin{aligned}
&(a)\ \widetilde{x}^i = L^i_j(t) x^j + S^i(y,z,t), \quad &&(b)\ \widetilde{y}^\lambda = \widetilde{y}^\lambda(y,t), \\
&(c)\ \widetilde{z}^\alpha = \widetilde{z}^\alpha(z,t), &&(d)\ \widetilde{t}^i = \widetilde{t}^i(t),
\end{aligned} \tag{7.6}$$

where S^i are smooth functions on $\mathcal{U} \cap \widetilde{\mathcal{U}}$. Next, denote by $A_{\lambda\mu}$ and $F_{\lambda i}$ the entries of the matrices $A(y,t)$ and $F(y,t)$ from (7.1). Hence we have

$$(a)\ A_{\lambda\mu} = g\left(\frac{\partial}{\partial y^\lambda}, \frac{\partial}{\partial y^\mu}\right) \quad \text{and} \quad (b)\ F_{\lambda i} = g\left(\frac{\partial}{\partial y^\lambda}, \frac{\partial}{\partial t^i}\right)\cdot \tag{7.7}$$

Then, by direct calculations using (7.7), (7.2b), (7.2d) and (7.1) we obtain

$$F_{\lambda i} = \frac{\partial \widetilde{x}^j}{\partial y^\lambda}\,\frac{\partial \widetilde{t}^k}{\partial t^i}\,\delta_{jk} + \frac{\partial \widetilde{y}^\mu}{\partial y^\lambda}\,\frac{\partial \widetilde{y}^\nu}{\partial t^i}\,\widetilde{A}_{\mu\nu} + \frac{\partial \widetilde{y}^\mu}{\partial y^\lambda}\,\frac{\partial \widetilde{t}^k}{\partial t^i}\,\widetilde{F}_{\mu k}. \tag{7.8}$$

By contracting (7.8) with $\frac{\partial t^i}{\partial \widetilde{t}^h}$ we deduce that $\frac{\partial \widetilde{x}^j}{\partial y^\lambda}$ are functions of (y^μ, t^k) alone. Then (7.6a) implies that $\frac{\partial S^i}{\partial y^\lambda}$ are functions of (y^μ, t^k) alone, and therefore S^i are written as follows:

$$S^i(y, z, t) = F^i(y, t) + G^i(z, t).$$

Hence (7.6) has the final form (cf. Robertson–Furness [RF74])

$$\begin{aligned} &(\text{a})\ \widetilde{x}^i = L^i_j(t)x^j + F^i(y,t) + G^i(z,t), \\ &(\text{b})\ \widetilde{y}^\lambda = \widetilde{y}^\lambda(y,t), \quad (\text{c})\ \widetilde{z}^i = \widetilde{z}^\alpha(z,t), \quad (\text{d})\ \widetilde{t}^i = \widetilde{t}^i(t). \end{aligned} \tag{7.9}$$

Moreover, comparing (7.9) with (5.24) we can state the following.

Theorem 7.2. (Robertson–Furness [RF74]). *Let $\mathcal{F}$ be a parallel partially–null foliation on a proper semi–Riemannian manifold (M, g). Then the totally–null foliation $\mathcal{F}_\mathcal{N}$ on M is a locally affine foliation.*

Taking into account that the canonical form (7.1) is preserved by the coordinate transformations (7.9), from Theorem 7.1 we deduce the following.

Theorem 7.3. *Let M be a $(2r + s + u)$–dimensional manifold that admits an atlas in which the change of coordinates is given by (7.9). If $\mathcal{F}$ is the $(r + s)$–foliation whose tangent distribution is locally represented by $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\lambda} \right\}$, $i \in \{1, ..., r\}$, $\lambda \in \{1, ..., s\}$, then there exists on M a proper semi–Riemannian metric g such that $\mathcal{F}$ is partially–null of type (r, s) and parallel with respect to the Levi–Civita connection on (M, g).*

Remark 7.1. It is easy to check that (7.1) is also preserved by the change or coordinates (7.6). Therefore Theorem 7.3 is still true when M admits an atlas whose change of coordinates is given by (7.6). ■

4.8 Manifolds with Walker Complementary Foliations

Given a distribution $\mathcal{D}$ on a manifold M, we saw in Chapter 1 the importance of using a complementary distribution $\mathcal{D}'$ for obtaining tools that help in understanding the geometry of the manifold. A good example of the importance of complementary distributions is the complete understanding of the global geometry of a semi–Riemannian manifold admitting a parallel non–degenerate distribution (where a natural complementary distribution exists) (see Section 4.4). The lack of global results for the partially–null case is due to the fact that, in general, such a "natural" complementary distribution does not exist.

In this section we study the geometry of proper semi–Riemannian manifolds admitting a parallel partially–null distribution and a natural complementary distribution. The emphasis will be on parallel totally–null r–foliations of $2r$–dimensional semi–Riemannian manifolds. But first, let us make the term "natural complementary" a specific one.

Let (M, g) be an m–dimensional proper semi–Riemannian manifold and $\mathcal{D}$ be a parallel partially–null distribution of type (r, s) (see Section 4.3). Hence $\mathcal{D}$ is an integrable distribution that is tangent to a parallel partially–null $(r+s)$–foliation $\mathcal{F}$. By Theorem 3.1, M admits three more parallel foliations $\mathcal{F}^{\perp}, \mathcal{F}^{+}$ and $\mathcal{F}_{\mathcal{N}}$ with tangent distributions $\mathcal{D}^{\perp}$, $\mathcal{D}^{+} = \mathcal{D} + \mathcal{D}^{\perp}$ and $\mathcal{N} = \mathcal{D} \cap \mathcal{D}^{\perp}$ respectively. We have seen in Section 4.7 that on $(M, g, \mathcal{F})$ there exists a Walker atlas $\mathcal{A}$ whose local coordinates $(x^i, y^\lambda, z^\alpha, t^i)$ are changed according to (7.9). Taking into account (7.2d) we deduce that $\left\{ \dfrac{\partial}{\partial t^1}, \dots, \dfrac{\partial}{\partial t^r} \right\}$ on the domain $\mathcal{U}$ of any local chart $(\mathcal{U}; \varphi)$ from $\mathcal{A}$ need not define a global distribution on M.

In this section we impose the additional condition that the local vector fields $\left\{ \dfrac{\partial}{\partial t^i} \right\}$, $i \in \{1, ..., r\}$, induced by a Walker atlas define a global distribution $\mathcal{D}^c$ on M. We call $\mathcal{D}^c$ the **Walker complementary distribution**. By using Theorem 1.1.1. and the definition of $\mathcal{D}^c$ we obtain the following.

Proposition 8.1. *The Walker complementary distribution is integrable.*

Thus we obtain a fifth foliation $\mathcal{F}^c$ whose tangent distribution is $\mathcal{D}^c$ and therefore it is complementary to $\mathcal{F}^{+}$. We call $\mathcal{F}^c$ the **Walker complementary foliation**. The next theorem states an interesting result on the local structure of $\mathcal{F}^c$.

Theorem 8.2. *Let (M, g) be a $(2r + s + u)$–dimensional proper semi–Riemannian manifold and $\mathcal{F}$ be a parallel partially–null foliation of type (r, s) on M. Suppose that M admits a Walker complementary foliation $\mathcal{F}^c$. Then $\mathcal{F}^c$ is a locally affine foliation.*

Proof. In the previous section we have seen that M admits a Walker atlas $\mathcal{A}$ in which the change of coordinates is given by (7.9). Since M also admits a Walker complementary foliation $\mathcal{F}^c$, the functions $\widetilde{x}^i$, $\widetilde{y}^\lambda$ and $\widetilde{z}^\alpha$ from (7.9) must be independent of $(t^1, ..., t^r)$. Thus from (7.9a) we deduce that $L^i_j(t)$ given by (7.5b) must be constant. Then (7.9) becomes

$$\begin{aligned} &\text{(a) } \widetilde{x}^i = b^i_j x^j + F^i(y) + G^i(z), \\ &\text{(b) } \widetilde{y}^\lambda = \widetilde{y}^\lambda(y), \quad \text{(c) } \widetilde{z}^\alpha = \widetilde{z}^\alpha(z), \quad \text{(d) } \widetilde{t}^i = a^i_j t^j + b^i, \end{aligned} \tag{8.1}$$

where a^i_j, b^i_j, b^i are constant and we have

$$[b^i_j] = ([a^i_j]^T)^{-1}. \tag{8.2}$$

Then our assertion follows from (8.1) by using Theorem 5.6. ∎

Corollary 8.3. *Let (M, g) be a $2r$–dimensional proper semi–Riemannian manifold and $\mathcal{F}$ be a parallel totally–null r–foliation on M. Suppose that M admits a Walker complementary foliation $\mathcal{F}^c$. Then M is a locally affine manifold.*

Proof. By the same arguments as in the proof of the above theorem, we deduce that the coordinate transformations (6.14) in a Walker atlas on M become

$$\begin{aligned} \widetilde{x}^i &= b^i_j x^j + c^i, \\ \widetilde{t}^i &= a^i_j t^j + b^i. \end{aligned} \tag{8.3}$$

Then the assertion follows from (8.3) by using Theorem 5.4. ∎

Now, combining Theorem 5.5 with Corollary 8.3, we state the following result on the global structure of (M, g).

Corollary 8.4. *Let $(M, g, \mathcal{F})$ be a foliated semi–Riemannian manifold as in Corollary 8.3. Then M is affinely equivalent to $(\mathbb{E}^{2r}, \widetilde{\nabla})/G$ where G is a properly discontinuous subgroup of the affine group $A(2r, \mathbb{R})$ that is isomorphic to $\Pi_1(M)$.*

Now, we want to relate complex structures on manifolds with parallel totally–null foliations on $2r$–dimensional semi–Riemannian manifolds. To this end we need some terminology. Let M be a complex manifold of complex dimension r. Then M can be considered as a real $2r$–dimensional manifold with local coordinates $(x^1, ..., x^r, t^1, ..., t^r)$ where $z^j = x^j + it^j$, $j \in \{1, ..., r\}$, are the local complex coordinates on M. Moreover, the coordinate transformations on M, given by

$$\widetilde{x}^i = \widetilde{x}^i(x^j, t^j), \quad \widetilde{t}^i = \widetilde{t}(x^j, t^j), \tag{8.4}$$

satisfy the Cauchy–Riemann equations:

$$\text{(a)} \ \ \frac{\partial \widetilde{x}^i}{\partial x^j} = \frac{\partial \widetilde{t}^i}{\partial t^j}, \quad \text{(b)} \ \ \frac{\partial \widetilde{x}^i}{\partial t^j} = -\frac{\partial \widetilde{t}^i}{\partial x^j}. \tag{8.5}$$

The above atlas of real charts on M is called a **Cauchy–Riemann atlas**. When M admits a semi–Riemannian metric g and a parallel totally–null foliation $\mathcal{F}$ whose leaves are locally given by t^i = constant, we say that M has a **Cauchy–Riemann atlas of Walker type**. Now, we can prove the following.

Theorem 8.5. *Let M be a complex manifold which admits a Cauchy–Riemann atlas of Walker type. Then M must be locally Euclidean.*

Proof. Since M admits the foliation $\mathcal{F}$, $\widetilde{t}^i$ from (8.4) must be independent of $(x^1, ..., x^r)$. Thus, by (8.5b), $\widetilde{x}^i$ from (8.4) must be independent of $(t^1, ..., t^r)$.

This means that M admits a Walker complementary foliation $\mathcal{F}^c$. Then, by Corollary 8.3, we deduce that M is a locally affine manifold. Moreover, by using (8.5a) and (8.3), we obtain $a^i_j = b^i_j$. Taking into account (8.2) we conclude that the transformations (8.3) are local isometries of a $2r$–dimensional semi–Euclidean space. Hence M is a locally Euclidean manifold. ■

In the last part of this section we show the existence of a Walker complementary foliation to the foliation by fibers on the cotangent bundle of a locally affine manifold. First we prove the following.

Proposition 8.6. *Let M be a locally affine r–manifold. Then the cotangent bundle T^*M admits a complementary foliation to that given by fibers.*

Proof. Let (t^i, x_i), $i \in \{1, ..., r\}$, be the local coordinates on T^*M, where (t^i) are the local coordinates on M. Then by (5.23) and (6.18) the transformations of coordinates on T^*M have the special form

$$\widetilde{t^i} = a^i_j t^j + b^i, \quad x_i = a^j_i \widetilde{x}_j, \quad i \in \{1, ..., r\}.$$

Thus on a coordinate neighbourhood in T^*M we have

$$\frac{\partial}{\partial t^i} = a^j_i \, \frac{\partial}{\partial \widetilde{t^j}} \, , \; i \in \{1, ..., r\}.$$

Hence there exists an integrable distribution on T^*M locally spanned by $\left\{ \frac{\partial}{\partial t^i} \right\}$, $i \in \{1, ..., r\}$. Clearly, the corresponding foliation is complementary to the foliation by fibers. ■

We note that the above proposition is a general one in the sense that the parallelism and nullity of the foliation by fibers were not mentioned. This enables us to obtain the following general corollary.

Corollary 8.7. *The cotangent bundle T^*M of a locally affine r–manifold M is diffeomorphic to $\mathbb{E}^{2r}/G$, where G is a subgroup of affine transformations of $\mathbb{E}^{2r}$ acting freely and properly discontinuously.*

Proof. Since M is locally affine, by Proposition 8.6 we see that T^*M admits a foliation complementary to that given by fibers. But Theorem 6.7 says that T^*M admits a proper semi–Riemannian metric such that the foliation by fibers is parallel and totally–null. As the atlas on T^*M with local coordinates (t^i, x_i) is a Walker atlas (see the proof of Theorem 6.7), we apply Corollary 8.4 and obtain our assertion. ■

Now, we recall from Section 4.6 that para–Kählerian manifolds provide examples of $2r$–dimensional semi–Riemannian manifolds that admit pairs of parallel and totally–null complementary foliations. We show here that cotangent bundles of locally affine manifolds are another good source of such examples. But first we recall the concept of Riemann extension introduced by Patterson–Walker [PW52]. Let ∇ be a torsion–free linear connection on an r–dimensional manifold M. Denote by (t^i, x_j) the local coordinates on T^*M and by $\Gamma_i{}^k{}_j$ the local coefficients of ∇ with respect to the local coordinates (t^i) on M. Then the matrix

$$[h] = \begin{bmatrix} -2x_k \Gamma_i{}^k{}_j & \delta_{ij} \\ \delta_{ij} & 0 \end{bmatrix}, \tag{8.6}$$

defines a global semi–Riemannian metric h on T^*M which is called a **Riemann extension**. Now, we can prove the following.

Theorem 8.8. *Let M be a locally affine r–dimensional manifold and T^*M the cotangent bundle of M. If $\mathcal{F}$ is the foliation of T^*M by fibers, then T^*M admits a foliation $\mathcal{F}^c$ complementary to $\mathcal{F}$ and a semi–Riemannian metric for which both $\mathcal{F}$ and $\mathcal{F}^c$ are parallel and totally–null.*

Proof. Since M is locally affine, by Proposition 8.6 a complementary foliation $\mathcal{F}^c$ to $\mathcal{F}$ exists on M. Also, on M there exists a torsion–free linear connection ∇ with vanishing curvature. Moreover, M admits local coordinates (t^i) with respect to which all the local coefficients of ∇ vanish on M. Then we consider the induced local coordinates (t^i, x_i) on T^*M and by using (8.6) we obtain a semi–Riemannian metric h on T^*M whose matrix is

$$[h] = \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix}. \tag{8.7}$$

Finally, comparing (8.7) with (6.13) and applying Theorem 6.4, we conclude that both $\mathcal{F}$ and $\mathcal{F}^c$ are parallel and totally–null with respect to h. ∎

4.9 Parallel Foliations and *G*–Structures

In Chapter 2 we presented different approaches to foliations. We discuss now yet another approach that was not mentioned there. This is the approach to foliations using G–structures. In particular, we obtain characterizations of parallel foliations in terms of G–structures.

The theory of G–structures was introduced by Chern [Che53] and plays a central role in differential geometry. Let us start by giving a brief introduction to the subject.

Let P be a manifold and G a Lie group. Suppose that G acts to the right as a Lie transformation group on P, i.e., there exists a smooth mapping $\Phi : P \times G \longrightarrow P$ satisfying the conditions (see Example 2.1.7)

(i) $\Phi(\Phi(p,a),b) = \Phi(p, a*b), \quad \forall\, a,b \in G,\ p \in P$, where $*$ is the operation on G.

(ii) $\Phi(p,e) = p$, $\forall\, p \in P$, where e is the unit element of G. Thus, for any $a \in G$ we have a diffeomorphism R_a of P onto itself given by $R_a(p) = \Phi(p,a)$. Next, we consider a manifold M and a smooth map π of P onto M. Then (P,M,π,G) is said to be a **principal bundle over M with structure group G** (shortly **principal G–bundle**) if the following conditions are satisfied (cf. Sternberg [Ste83], p. 294).

(a) G acts freely on P, i.e., for any $p \in P$ if $R_a(p) = p$, then $a = e$.

(b) Let p and p' be any two points of P. Then $\pi(p) = \pi(p')$ if and only if there is an $a \in G$ such that $R_a(p) = p'$. Thus M can be thought of (via π) as a quotient space of P under the action of G.

(c) P is locally trivial over M, that is, any $x \in M$ has a neighbourhood $\mathcal{U}$ and a diffeomorphism $\Psi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}\times G$ such that $\Psi(p) = (\pi(p), \varphi(p))$ and $\Psi(R_a(p)) = (\pi(p), \varphi(p)*a)$. According to the condition (c) we can choose an open covering $\{\mathcal{U}_\alpha\}$ of M such that $\Psi_\alpha(p) = (\pi(p), \varphi_\alpha(p))$ are diffeomorphisms of $\pi^{-1}(\mathcal{U}_\alpha)$ onto $\mathcal{U}_\alpha\times G$ and $\varphi_\alpha(R_a(p)) = \varphi_\alpha(p)*a$. Then for any $p \in \pi^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ we have

$$\varphi_\beta(R_a(p)) * (\varphi_\alpha(R_a(p)))^{-1} = \varphi_\beta(p) * (\varphi_\alpha(p))^{-1}.$$

Thus the map $p \longrightarrow \varphi_\beta(p) * (\varphi_\alpha(p))^{-1}$ is constant along fibers over $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. This enables us to define the map

$$\Psi_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow G, \quad \Psi_{\beta\alpha}(x) = \varphi_\beta(p) * (\varphi_\alpha(p))^{-1}, \tag{9.1}$$

where p is any point of $\pi^{-1}(x)$. The maps $\Psi_{\beta\alpha}$ are called the **transition functions** of the principal bundle (P,M,π,G) with respect to the covering $\{\mathcal{U}_\alpha\}$ of M. By using (9.1) it is easy to check that the transition functions satisfy

$$\Psi_{\gamma\beta} * \Psi_{\beta\alpha} = \Psi_{\gamma\alpha}. \tag{9.2}$$

It is important to note that a principal bundle can be constructed by using some functions $\Psi_{\beta\alpha}$ satisfying (9.2). More precisely, the following proposition is proved in Kobayashi–Nomizu [KN63], p. 52.

Proposition 9.1. *Let M be a manifold, $\{\mathcal{U}_\alpha\}$ an open covering of M and G a Lie group. Given a mapping $\Psi_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow G$ for every non–empty $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, in such a way that* (9.2) *is satisfied, we can construct a principal fiber bundle (P,M,π,G) with transition functions $\Psi_{\beta\alpha}$.*

Next, let (P_2, M, π_2, G_2) be a principal G_2–bundle over M and G_1 a Lie subgroup of G_2. Then it is said that P_2 has a **reduction** to a G_1–bundle (P_1, M, π_1, G_1) if there exists a smooth map $f : P_1 \longrightarrow P_2$ satisfying

$$f(R_a(p_1)) = R_a(f(p_1)), \quad \forall\, p_1 \in P_1 \ \text{ and } \ \forall\, a \in G_1.$$

Also we say that P_2 is **reducible** to the subgroup G_1, if there exists a reduction of P_2 to a G_1–bundle P_1. The following theorem is well known (see Sternberg [Ste83], p. 296 for a proof).

Theorem 9.2. *Let P_2 be a principal G_2–bundle over M and G_1 be a Lie subgroup of G_2. Then P_2 has a reduction to a principal G_1–bundle if and only if there is a covering of M whose transition functions take their values in G_1.*

The bundle of linear frames over a manifold has a great role in studying G–structures and linear connections. We present it here as an example of a principal bundle. Let M be an m–dimensional manifold and $L(M)$ be the set of all $(m+1)$–tuples $(x; E_1, ..., E_m)$, where $x \in M$ and $(E_1, ..., E_m)$ is a basis of T_xM which is called a **linear frame** at x. If $\{e_1 = (1, 0, ..., 0), ..., e_m = (0, ..., 0, 1)\}$ is the natural basis for $\mathbb{R}^m$, then a linear frame $(E_1, ..., E_m)$ at x can be thought of as a linear map $p : \mathbb{R}^m \longrightarrow T_xM$ such that $p(e_i) = E_i$, $i \in \{1, ..., m\}$. The general linear group $GL(m; \mathbb{R})$ of all non–singular $m \times m$ matrices acts to the right on $L(M)$ as follows. If $(x; E_1, ..., E_m) \in L(M)$ then

$$R_a(x; E_1, ..., E_m) = (x; a_1^i E_i, ..., a_m^i E_i),$$

where $a = [a_j^i] \in GL(m; \mathbb{R})$. Now, let $\{(\mathcal{U}, \eta) : (x^1, ..., x^m)\}$ be a local chart about a point $x \in M$. Then any vector of the linear frame $(E_1, ..., E_m)$ can be expressed as follows

$$E_i = E_i^j \left.\frac{\partial}{\partial x^j}\right|_x, \quad i \in \{1, ..., m\}. \tag{9.3}$$

Denote by $\pi : L(M) \longrightarrow M$ the natural projection, that is, $\pi(x; E_1, ..., E_m) = x$, and define (x^j, E_i^j) as local coordinates in $\pi^{-1}(\mathcal{U}) \subset L(M)$. Thus $L(M)$ becomes an $m(m+1)$–dimensional smooth manifold. Moreover it is easy to check that $(L(M), M, \pi, GL(m; \mathbb{R}))$ is a principal bundle. Finally, we note that $L(M)$ is known under the name **bundle of linear frames** over M.

Now, let G be a Lie subgroup of $GL(m; \mathbb{R})$. Then a G–**structure** on M is a reduction of the bundle of linear frames $L(M)$ to a principal G–bundle. Thus a G–structure on M is a submanifold S_G of $L(M)$ with the property that for any $p \in S_G$ and any $a \in GL(m; \mathbb{R})$ the point $R_a(p)$ belongs to S_G if and only if $a \in G$. Moreover, from Theorem 9.2 we immediately obtain the following.

Corollary 9.3. *Let M be an m–dimensional manifold and G a Lie subgroup of $GL(m; \mathbb{R})$. Then M admits a G–structure if and only if there is a covering of M whose transition functions take their values in G.*

The importance of G–structures comes from the fact that various geometric structures on a manifold M are reflected as G–structures, and conversely, if G is a Lie subgroup of $GL(m;\mathbb{R})$, then a G–structure on M has its geometric interpretation. Moreover, the existence of a G–structure on M is closely related to the geometry and topology of M. For example, if $G = \{e\}$ is the identity subgroup of $GL(m;\mathbb{R})$, then a G–structure on M defines a linear frame $(E_1, ..., E_m)$ at each point $x \in M$. We therefore have a family of m independent vector fields globally defined on M. For this reason it is said that an $\{e\}$–structure determines a **parallelization** on M, or M is a **parallelizable manifold**. In this case the tangent bundle TM is trivial, i.e., it is diffeomorphic to $M \times \mathbb{R}^m$. Any Lie group is a parallelizable manifold with a parallelization given by the left invariant vector fields. Also, the spheres S^1, S^3 and S^7 are parallelizable (see Brickell–Clark [BC70], p. 117).

We shall see later on in this section that Riemannian (semi–Riemannian) structures, distributions and foliations can be defined in terms of G–structures. Another example is when G represents the general linear complex group $GL(n;C)$, embedded as a subgroup of $GL(2n;\mathbb{R})$ in a natural way. In this case a G–structure on a real $2n$–dimensional manifold is nothing but an almost complex structure on M (see Example 2.1.8). We can describe, in a similar way, almost Hermitian structures, almost symplectic structures, conformal structures, etc., as G–structures with the corresponding subgroups G of $GL(m;\mathbb{R})$. More examples and results on the theory of G–structures can be found in Bernard [Ber60], Chern [Che66], Fujimoto [Fuj60] and in Chapter VII of Sternberg's book [Ste83].

Now, to describe foliations on Riemannian (semi–Riemannian) manifolds using G–structures, it might be useful to start with some elementary linear algebra. Let n and p be two positive integers and $m = n+p$. As in Section 2.1 we identify $\mathbb{R}^m$ with $\mathbb{R}^n \times \mathbb{R}^p$, and let $a, b, c, ... \in \{1, ..., m\}$, $i, j, k, ... \in \{1, ..., n\}$ and $\alpha, \beta, \gamma, ... \in \{n+1, .., n+p\}$. Consider the natural basis $\{e_1, ..., e_m\}$ of $\mathbb{R}^m$, where $e_1 = (1, 0, ..., 0), ..., e_m = (0, ..., 0, 1)$. Using this basis, the group of all linear isomorphisms of $\mathbb{R}^m$ is identified with $GL(m;\mathbb{R})$. Now if we look at $\mathbb{R}^n$ as a subspace of $\mathbb{R}^m$, then the subgroup G of all linear isomorphisms of $\mathbb{R}^m$ that leave $\mathbb{R}^n$ invariant is identified with the group of all non–singular $m \times m$ matrices of the form

$$\begin{bmatrix} A_{ij} & B_{i\beta} \\ 0 & C_{\alpha\beta} \end{bmatrix}. \tag{9.4}$$

Next, we consider an integer $0 \leq r \leq m$ and define the **pseudo–orthogonal group** $O(m;r)$ as follows

$$O(m;r) = \{A \in GL(m;\mathbb{R}) : A^T I_{(r,m-r)} A = I_{(r,m-r)}\}, \tag{9.5}$$

where we put

$$I_{(r,m-r)} = \begin{bmatrix} I_{m-r} & 0 \\ 0 & -I_r \end{bmatrix},$$

and I_s is the identity $s \times s$ matrix. In particular, for $r = 0$ we obtain the **orthogonal group**

$$O(m) = \{A \in GL(m; \mathbb{R}) : A^T A = I_m\}. \tag{9.6}$$

Now, we are in a position to present distributions, foliations and Riemannian (semi–Riemannian) structures by using G–structures.

Theorem 9.4. *Let M be a real m–dimensional manifold and G the group of all non–singular matrices of the form* (9.4)*. Then M admits an n–distribution if and only if M admits a G–structure.*

Proof. Let $\mathcal{D}$ be an n–distribution on M. Then for any $x \in M$ we denote by S_x the set of all linear frames $(E_1, ..., E_n, E_{n+1}, ..., E_{n+p})$ at x such that $\{E_1, ..., E_n\}$ spans $\mathcal{D}_x$. Thus, taking into account the form of matrices in G given by (9.4), we conclude that $S = \bigcup_{x \in M} S_x$ is a G–structure on M. Indeed, for any $p = (x; E_1, ..., E_m) \in S$ and any $a \in GL(m; \mathbb{R})$ we have $R_a(p) \in S$ if and only if $a \in G$. Conversely, let S_G be a G–structure on M with G given by (9.4). Then for any $x \in M$ we take $p = (x; E_1, ..., E_n, E_{n+1}, ..., E_{n+p}) \in S_G$ and define $\mathcal{D}_x$ as the subspace of T_xM spanned by $\{E_1, ..., E_n\}$. Now, $\mathcal{D}_x$ is independent of the choice of p. Indeed, if $q = (x; F_1, .., F_n, F_{n+1}, ..., F_{n+p}) \in S_G$ then there exists $a \in GL(m; \mathbb{R})$ such that $q = R_a(p)$. Since both $p, q \in S_G$, then $a \in G$. Thus if $\mathcal{D}'_x = \text{span}\{F_1, ..., F_n\}$, then $\mathcal{D}'_x = R_g(\mathcal{D}_x) = \mathcal{D}_x$ since the action of G leaves $\mathcal{D}_x$ invariant. This shows that M admits an n–distribution. ∎

Now, we recall from Section 1.1 that M has an almost product structure if and only if M admits two complementary distributions. Then from Theorem 9.4 we deduce the following.

Corollary 9.5. *An m–dimensional manifold M, $m > 1$, admits an almost product structure if and only if there exists a positive integer $n < m$ such that M admits a G–structure, where G is the subgroup of $GL(m; \mathbb{R})$ of matrices of the form*

$$\begin{bmatrix} A_{ij} & 0 \\ 0 & B_{\alpha\beta} \end{bmatrix}, \quad \begin{array}{l} i, j \in \{1, ..., n\} \\ \alpha, \beta \in \{n+1, ..., m\}. \end{array} \tag{9.7}$$

Next, to characterize foliations by using G–structures, we need to introduce a particular class of G–structures. Let S_G be a G–structure on an m–dimensional manifold M. A local chart $\{(\mathcal{U}, \varphi) : (x^1, ..., x^m)\}$ in M is said to be **admissible** with respect to S_G if the frame field $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right\}$ is a section of S_G over $\mathcal{U}$. A G-structure S_G on M is called **integrable** if M admits an atlas $\mathcal{A}$ of admissible charts. Now, combining Theorems 1.1.1, 2.1.1 and 9.4 we obtain the following.

Theorem 9.6. *Let M be an $(n+p)$–dimensional manifold with $n > 0$, $p > 0$. Then M admits an n–foliation if and only if it admits an integrable G–structure, where G is given by* (9.4).

As a consequence of Theorem 9.6 and Corollary 9.5 we can state the following.

Corollary 9.7. *An m–dimensional manifold M, $m > 1$, admits a pair of complementary foliations if and only if there exists a positive integer $n < m$ such that M admits an integrable G–structure, where G is given by* (9.7).

Now, we present Riemannian (semi–Riemannian) manifolds by using the theory of G–structures where G is the orthogonal group (pseudo–orthogonal group). First we prove the following.

Proposition 9.8. *Let M be an m–dimensional manifold. Then any $O(m)$–structure on M gives rise to a Riemannian metric on M. Conversely, any Riemannian metric on M defines an $O(m)$–structure on M.*

Proof. Let S be an $O(m)$–structure on M and $h : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be the Euclidean inner product on $\mathbb{R}^m$ (see (1.4.11)). Take $x \in M$ and $p \in \pi^{-1}(x)$, where $\pi : S \longrightarrow M$ is the projection map. Then we define the map

$$g_x : T_xM \times T_xM \longrightarrow \mathbb{R}; \quad g_x(u, v) = h(p^{-1}(u), p^{-1}(v)),$$

where we consider p as a linear isomorphism from $\mathbb{R}^m$ onto T_xM. First we note that g_x is independent of the choice of p since h is invariant under the action of $O(m)$. Then we see that g_x is positive definite and symmetric bilinear map, because h is so. Thus the map $g : x \longrightarrow g_x$ is a Riemannian metric on M. Conversely, let g be a Riemannian metric on M. Then for any $x \in M$ we define S_x as the set of $(m+1)$–tuples $(x; E_1, ..., E_m)$, where $(E_1, ..., E_m)$ is an orthonormal basis with respect to g_x. Then $S = \bigcup_{x \in M} S_x$ is an $O(m)$–structure. Indeed, for any $p \in S$ and $a \in GL(m; \mathbb{R})$, $R_a(p) \in S$ if and only if $a \in O(m)$. This is because the transition matrix between two orthonormal bases must be an orthogonal matrix. Thus the proof is complete. ■

Corollary 9.9. *Let K and L be two manifolds of dimensions k and ℓ respectively, and $M = K \times L$. Then a G–structure S_G on M defines a product Riemannian metric $g = h \times \lambda$, where h and λ are Riemannian metrics on K and L respectively, if and only if $G = O(k) \times O(\ell)$.*

Theorem 9.10. *Let $(M, \widetilde{g})$ be a Riemannian m–dimensional manifold and n, p be two positive integers such that $m = n + p$. Then $(M, \widetilde{g})$ admits a parallel n–foliation with respect to the Levi–Civita connection if and only if it admits an integrable G–structure, where G is the subgroup of $GL(m; \mathbb{R})$ given by* (9.7) *with $[A_{ij}] \in O(n)$ and $[B_{\alpha\beta}] \in O(p)$.*

Proof. Let $\mathcal{F}$ be a parallel n–foliation of $(M, \widetilde{g})$. Using Theorem 4.2 we conclude that any point $x \in M$ has a coordinate neighbourhood $\widetilde{\mathcal{V}}$ such that $(\widetilde{\mathcal{V}}, \widetilde{g}) = (\mathcal{V}, g) \times (\mathcal{V}^\perp, g^\perp)$, where $(\mathcal{V}, g)$ and $(\mathcal{V}^\perp, g^\perp)$ are Riemannian submanifolds of $(M, \widetilde{g})$ of dimensions n and p respectively. Since $\widetilde{g} = g \times g^\perp$ (see

(4.1)), then using Corollary 9.9 we conclude that the G–structure defined by $\widetilde{g}$ must have $G = O(n)\times O(p)$. This says that elements of G are of the form (9.7) with $[A_{ij}] \in O(n)$ and $[B_{\alpha\beta}] \in O(p)$. Conversely, suppose that $(M, \widetilde{g})$ admits an integrable G–structure with G as in the theorem. Then it follows from Corollary 9.7 that $(M, \widetilde{g})$ admits a pair of complementary foliations $\mathcal{F}$ and $\mathcal{F}'$ of codimensions p and n respectively. Using Theorem 2.2 we deduce that for every $x \in M$, there is a coordinate neighbourhood $\widetilde{\mathcal{V}} = \mathcal{V}\times\mathcal{V}'$, where $\mathcal{V}$ and $\mathcal{V}'$ are open submanifolds of leaves of $\mathcal{F}$ and $\mathcal{F}'$ through x. Then using Corollary 9.9 again, we conclude that $(\widetilde{\mathcal{V}}, \widetilde{g})$ is a Riemannian product $(\mathcal{V}, g)\times(\mathcal{V}', g')$ (see (2.4)). Thus $(\mathcal{V}, g)$ and $(\mathcal{V}', g')$ are totally geodesic immersed in $(\widetilde{\mathcal{V}}, \widetilde{g})$, and by Theorem 4.3 the foliations $\mathcal{F}$ and $\mathcal{F}'$ ar parallel and mutually orthogonal with respect to $\widetilde{g}$. ∎

Now, we note that Proposition 9.8 can be extended to semi–Riemannian manifolds. That is, the existence of an $O(m, r)$–structure on M is equivalent to the existence of a semi–Riemannian metric of index r on M, where $O(m; r)$ is the pseudo–orthogonal group given by (9.5). Moreover, a slightly modified version of Theorem 9.10 is still true for parallel non–degenerate foliations on semi–Riemannian manifolds. This is because Theorems 4.2 and 4.3 still apply to this case. To be more specific, we give the following theorem, whose proof is similar to that of Theorem 9.10 and will be omitted here.

Theorem 9.11. *Let $(M, \widetilde{g})$ be a semi–Riemannian m–dimensional manifold of index r, and n, p be two positive integers such that $m = n + p$. Then $(M, \widetilde{g})$ admits a parallel non–degenerate n–foliation if and only if it admits a G–structure, where G is the subgroup of $GL(m; \mathbb{R})$ given by (9.7) with $[A_{ij}] \in O(n; s)$ and $[B_{\alpha\beta}] \in O(p; t)$ for some non–negative integers s, t with $s + t = r$.*

We go now to study parallel partially–null foliations by using G–structures. The case of parallel totally–null foliations will be obtained as a special case.

From now on, in this section, r will be a positive integer, s and u are non–negative integers and $m = 2r + s + u$. Let $W(m, r, s)$ be the collection of all elements of $GL(m; \mathbb{R})$ of the form

$$\begin{bmatrix} (A_{11}^{-1})^T & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{11} \end{bmatrix}, \tag{9.8}$$

where A_{11}, A_{22} and A_{33} are non–singular $r\times r$, $s\times s$ and $u\times u$ matrices respectively. The order of the other submatrices is determined accordingly. It is easy to check that $W(m, r, s)$ is a Lie subgroup of $GL(m; \mathbb{R})$. Moreover we have the following characterization of parallel degenerate foliations.

Theorem 9.12. (Farran [Far80]). *If an m–dimensional semi–Riemannian manifold (M, g) admits a foliation $\mathcal{F}$ of type (r, s), then it admits an inte-*

grable $W(m,r,s)$–structure. Conversely, an m–dimensional manifold M with an integrable $W(m,r,s)$–structure admits a foliation $\mathcal{F}$ and a semi–Riemannian metric g such that $\mathcal{F}$ is parallel of type (r,s) with respect to g.

Proof. Suppose that the manifold $(M,\widetilde{g})$ admits a parallel foliation $\mathcal{F}$ of type (r,s). Then M also admits three more parallel foliations $\mathcal{F}^{\perp},\mathcal{F}^{+}$ and $\mathcal{F}_{\mathcal{N}}$ of type (r,u), $(r,s+u)$ and $(r,0)$ respectively, where r,s,u verify (3.2) (see Section 4.7). Using Theorem 7.1 we conclude that M admits a Walker atlas $\mathcal{A}$ in which the change of coordinates takes the form (7.9). Now, we consider the covering of M by coordinate neighbourhoods $\{\mathcal{U}_\alpha\}$ of $\mathcal{A}$ and define the transition functions

$$\psi_{\beta\alpha}:\mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow GL(m;\mathbb{R}),\quad \psi_{\beta\alpha}(x)=J_{\beta\alpha}(x),$$

where $[J_{\beta\alpha}(x)]$ is the Jacobian matrix of the transformation of coordinates (7.9). It is easy to see that $\psi_{\beta\alpha}$ take all their values in $W(m,r,s)$ and hence by Corollary 9.3 we conclude that M admits a $W(m,r,s)$–structure. Since $\mathcal{A}$ is an atlas with admissible local charts with respect to this structure, it follows that the $W(m,r,s)$–structure is integrable.

Conversely, suppose that M admits an integrable $W(m,r,s)$–structure. Then consider the decomposition $\mathbb{R}^m=\mathbb{R}^r\times\mathbb{R}^s\times\mathbb{R}^u\times\mathbb{R}^r$ and take on M an atlas $\mathcal{A}$ with local coordinates $(x^i,y^\lambda,z^\alpha,t^j)$ such that the local natural field of frames $\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^\lambda},\frac{\partial}{\partial z^\alpha},\frac{\partial}{\partial t^j}\right\}$ belongs to the $W(m,r,s)$–structure. This is possible because the $W(m,r,s)$–structure is supposed to be integrable. We use here the same range of indices as in Section 4.7, that is: $i,j,k,...\in\{1,...,r\}$; $\alpha,\beta,\gamma,...\in\{1,...,u\}$ and $\lambda,\mu,\nu,...\in\{1,...,s\}$. Taking into account the zero submatrices in (9.8) we deduce that the entries of the Jacobian matrix of the transformation of coordinates in $\mathcal{A}$ should satisfy the following:

$$\frac{\partial\widetilde{y}^\lambda}{\partial x^i}=\frac{\partial\widetilde{z}^\alpha}{\partial x^i}=\frac{\partial\widetilde{t}^j}{\partial x^i}=0,$$
$$\frac{\partial\widetilde{z}^\alpha}{\partial y^\lambda}=\frac{\partial\widetilde{t}^j}{\partial y^\lambda}=0,$$
$$\frac{\partial\widetilde{y}^\lambda}{\partial z^\alpha}=\frac{\partial\widetilde{t}^j}{\partial z^\alpha}=0.$$

Hence the change of coordinates in $\mathcal{A}$ must be of the form

$$\begin{array}{ll}\text{(a) } \widetilde{x}^i=\widetilde{x}^i(x,y,z,t), & \text{(b) } \widetilde{y}^\lambda=\widetilde{y}^\lambda(y,t),\\ \text{(c) } \widetilde{z}^\alpha=\widetilde{z}^\alpha(z,t), & \text{(c) } \widetilde{t}^j=\widetilde{t}^j(t).\end{array}\tag{9.9}$$

Thus M admits four foliations $\mathcal{F}_x,\mathcal{F}_{xy},\mathcal{F}_{xz}$ and $\mathcal{F}_{xyz}$ whose tangent distributions are locally spanned by $\left\{\frac{\partial}{\partial x^i}\right\}$, $\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^\lambda}\right\}$, $\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial z^\alpha}\right\}$, and

$\left\{ \dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^\lambda}, \dfrac{\partial}{\partial z^\alpha} \right\}$ respectively. Moreover, since the first submatrix on the main diagonal in (9.8) is the transpose of the inverse of A_{11}, we conclude that

$$\frac{\partial \widetilde{x}^i}{\partial x^j} = L^i_j(t), \quad \text{where } L^i_j(t) = \frac{\partial t^j}{\partial \widetilde{t}^i} \cdot$$

Thus (9.9) becomes (7.6), and therefore by Remark 7.1 we conclude that there exists a semi–Riemannian metric g on M such that the foliation $\mathcal{F}_{xy}$ is parallel and partially–null of type (r, s) with respect to g. Finally, by using Theorems 5.3 and 6.5 we obtain the same conclusion for $s = 0$, $u > 0$ and for $s = u = 0$ respectively. This completes the proof of the theorem. ∎

Notice that in the above proof we have assumed that r is positive, but s and u were only assumed non–negative. Thus the result stated in Theorem 9.12 is general and takes care of all the following cases:

(i) Parallel totally–null r–foliations of $2r$–dimensional semi–Riemannian manifolds are obtained when $s = 0$ and $u = 0$.
(ii) Parallel totally–null r–foliations of m–dimensional semi–Riemannian manifolds are obtained when $s = 0$ and $u > 0$.
(iii) Parallel partially–null r–foliations for $s > 0$.

Finally, we remark that the case (iii) includes the special case when $u = 0$. In this case we have only two distinct parallel foliations, namely $\mathcal{F}_x$ and $\mathcal{F}_{xy}$. This is because $\mathcal{F}_{xz}$ coincides with $\mathcal{F}_x$ and $\mathcal{F}_{xyz}$ with $\mathcal{F}_{xy}$.

5

FOLIATIONS INDUCED BY GEOMETRIC STRUCTURES

This chapter deals with some interesting areas of interaction between the theory of foliations and several geometric structures. We will see that certain geometric structures on manifolds give rise to families of foliations on these manifolds in a natural way. Moreover, there is a strong relationship between the geometry of the manifold and that of the foliation.

The first section deals with Lagrange foliations on symplectic manifolds. We give Weinstein's results on the local structure of symplectic manifolds with Lagrange foliations. Also, we show a relationship between Lagrange foliations on symplectic manifolds and totally–null r–foliations on $2r$–dimensional semi–Riemannian manifolds (cf. Farran [Far79]).

Section 5.2 discusses Legendre foliations on contact manifolds. Following Pang [Pan90] and Libermann [Lib91] we present the local structure of both the Legendre foliations and the contact manifolds which carry such foliations. We also give some of the main results on the geometry of Legendre foliations on contact metric manifolds (cf. Jayne [Jay92], [Jay94]).

In Section 5.3 we investigate many natural foliations on the tangent bundle of a Finsler manifold. We show that information about these foliations can be interpreted as information about the Finsler structure and vice versa. It is noteworthy that the Vrănceanu connection which comes from the geometry of foliations (or, more generally, from the geometry of non–holonomic manifolds) incorporates all the classical connections from Finsler geometry: Berwald connection, Cartan connection, Rund connection and Hashiguchi connection. This new approach of Finsler geometry might help in solving some difficult problems in this field.

In the last section, following the general trend of this chapter, we investigate the relationship between the geometry of the totally real foliation on a CR–submanifold of a Kähler manifold and the geometry of the CR–submanifold itself. The section ends with results on the geometry of a CR–submanifold when its totally real foliation admits a complementary orthogonal complex foliation.

5.1 Lagrange Foliations on Symplectic Manifolds

We start the section with a brief presentation of the basic notions and results we need about symplectic vector spaces and symplectic manifolds.

Let V be a real m–dimensional vector space and $\Omega : V \times V \to \mathbb{R}$ be a **symplectic form** on V, that is Ω is a skew–symmetric non–degenerate bilinear form on V. Thus we have $\Omega(u, v) + \Omega(v, u) = 0$ for any $u, v \in V$ and if $\Omega(u, v) = 0$ for all $v \in V$, then $u = 0$. It follows that m must be even, and from now on we take $m = 2n$. A vector space V endowed with a symplectic form Ω is denoted by (V, Ω) and it is called a **symplectic vector space**. A basis $e = \{e_1, ..., e_{2n}\}$ can be chosen in (V, Ω) such that

$$\Omega = e_1^* \wedge e_{n+1}^* + \cdots + e_n^* \wedge e_{2n}^*, \tag{1.1}$$

where $e^* = \{e_1^*, ..., e_{2n}^*\}$ is the dual basis to e and $\wedge$ represents the exterior product on the dual space V^* of V.

Example 1.1. Let $\mathbb{R}^{2n}$ be equipped with the natural basis $e = \{e_1, ..., e_{2n}\}$. Then we define the bilinear map

$$\Omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}, \ \Omega(u, v) = \sum_{i=1}^{n} \{u^{n+i} v^i - u^i v^{n+i}\}, \tag{1.2}$$

where we put

$$u = \sum_{i=1}^{n} \{u^i e_i + u^{n+i} e_{n+i}\}, \quad v = \sum_{i=1}^{n} \{v^i e_i + v^{n+i} e_{n+i}\}.$$

It is easy to check that Ω is a symplectic form on $\mathbb{R}^{2n}$. The symplectic vector space $(\mathbb{R}^{2n}, \Omega)$ with Ω given by (1.2) is known as the **standard symplectic space**. ∎

Next, let (V, Ω) and (V', Ω') be two symplectic spaces of dimensions $2n$ and $2n'$ respectively. Then a linear map $L : V \longrightarrow V'$ is called **symplectic** if

$$\Omega'(Lu, Lv) = \Omega(u, v), \quad \forall\, u, v \in V. \tag{1.3}$$

Taking into account that Ω is non–degenerate we deduce that a symplectic linear map is injective and therefore $n \leq n'$. Thus, for $n = n'$, L must be an isomorphism of vector spaces. A symplectic isomorphism is called **symplectomorphism**. In particular, any symplectic linear map $L : (V, \Omega) \longrightarrow (V, \Omega)$ is necessarily an automorphism of (V, Ω). The set $Sp(V)$ of all symplectic linear maps of (V, Ω) is a group with respect to the usual composition. $Sp(V)$ is called the **symplectic group** of (V, Ω). In particular, when $V = \mathbb{R}^{2n}$ and Ω is given by (1.2) the symplectic group will be denoted $Sp(2n; \mathbb{R})$. To see the form of matrices in $Sp(2n; \mathbb{R})$ we put

$$S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix, and denote by U and V the column matrices whose entries are the components of vectors u and v respectively. Then the symplectic form Ω given by (1.2) is written in the matrix form as follows

$$\Omega(u, v) = U^T S V, \tag{1.4}$$

where U^T is the transpose of U. By using (1.3) and (1.4) we deduce that $A \in Sp(2n; \mathbb{R})$ if and only if

$$A^T S A = S.$$

Two vectors u and v in a symplectic space (V, Ω) are called Ω–**orthogonal** or **skew–orthogonal** if $\Omega(u, v) = 0$. Since Ω is skew–symmetric, then every vector $u \in (V, \Omega)$ is self Ω–orthogonal since $\Omega(u, u) = 0$.

Now, let W be a p–dimensional subspace of a $2n$–dimensional symplectic space (V, Ω). Then define the Ω–**orthogonal space** to W by

$$W^{\perp} = \{u \in V : \Omega(u, w) = 0 \text{ for all } w \in W\}. \tag{1.5}$$

The usual properties we met in the theory of semi–Euclidean geometry (see Section 1.4) are also true here, that is, we have

$$\text{(a)} \ \dim V = \dim W + \dim W^{\perp}, \quad \text{(b)} \ (W^{\perp})^{\perp} = W. \tag{1.6}$$

Also, we define the **radical** of W as the subspace $\operatorname{rad} W$ of W given by

$$\operatorname{rad} W = W \cap W^{\perp}. \tag{1.7}$$

Denote by $\Omega_{|W}$ the restriction of Ω to W and suppose that the rank of $\Omega_{|W}$ is $2q$. Then we have

$$\dim W = \dim \operatorname{rad} W + 2q. \tag{1.8}$$

By using W and $W^{\perp}$ we may consider the subspace

$$W^{+} = W + W^{\perp}, \tag{1.9}$$

whose dimension is given by

$$\dim W^{+} = \dim W + \dim W^{\perp} - \dim \operatorname{rad} W = 2n - p + 2q. \tag{1.10}$$

A symplectic space (V, Ω) has some interesting subspaces W. These subspaces are determined according to the behaviour of Ω on W. We describe in what follows some of these subspaces. First, we say that W is **non–degenerate** (resp. **degenerate**) if $\Omega_{|W}$ is non–degenerate (resp. degenerate). Clearly, $(W, \Omega_{|W})$ is a symplectic vector space when W is non–degenerate. By using (1.8) we can state the following:

Proposition 1.1. *Let W be a p–dimensional subspace of a symplectic vector space (V, Ω) and $2q = \operatorname{rank} \Omega_{|W}$. Then we have the assertions:*

(i) *W is non–degenerate if and only if* $p = 2q$, *or equivalently* $\operatorname{rad} W = \{0\}$.
(ii) *W is degenerate if and only if* $p > 2q$, *or equivalently* $\operatorname{rad} W \neq \{0\}$.

Next, let W be a p–dimensional degenerate subspace of a $2n$–dimensional symplectic space (V, Ω). Then $W \neq \{0\}$ and $W^{\perp} \neq \{0\}$, that is, $2n > p > 0$. We say that W is **isotropic** (**coisotropic**) when $W \subset W^{\perp}$ (resp. $W^{\perp} \subset W$). If W is both isotropic and coisotropic, that is $W = W^{\perp}$, we say that it is a **Lagrangian subspace**. Taking into account (1.5)–(1.7) we obtain the following characterizations.

Proposition 1.2. *Let W be a degenerate subspace of a symplectic space (V, Ω). Then we have the assertions:*

(i) *W is isotropic if and only if* $\Omega_{|W} = 0$.
(ii) *W is coisotropic if and only if* $\Omega_{|W^{\perp}} = 0$.
(iii) *W is Lagrangian if and only if* $\Omega_{|W^{+}} = 0$.

If (V, Ω) is a $2n$–dimensional symplectic space, then all its Lagrangian subspaces must be of dimension n. This follows immediately from (1.6a). Now, let W be a given Lagrangian subspace of (V, Ω). Then, following the idea from the proof of Theorem 4.6.1, using Ω instead of g, we can find a complementary Lagrangian subspace W^t to W in V. Thus we have

$$V = W \oplus W^t, \tag{1.11}$$

where W and W^t are both Lagrangian subspaces of the same dimension n. W^t is called a **transversal Lagrangian subspace** to W. Moreover, applying the construction for vector fields given by (4.6.5) to the symplectic case, we obtain a basis $\{e_i, e_{n+i}\}$ in (V, Ω) such that $\{e_i\}$ and $\{e_{n+i}\}$ are bases in W and W^t respectively, and satisfy

$$\begin{aligned} &\text{(a) } \Omega(e_i, e_j) = \Omega(e_{n+i}, e_{n+j}) = 0, \\ &\text{(b) } \Omega(e_i, e_{n+j}) = \delta_{ij}. \end{aligned} \tag{1.12}$$

Clearly, W^t given in (1.11) is not unique. Indeed, it is easy to check that $\operatorname{span}\{e_{n+1} + e_1, ..., e_{2n} + e_n\}$ is another transversal Lagrangian subspace to W. More about symplectic algebra can be seen in Artin [Art75] and Berndt [Ber01].

Now we extend the above symplectic algebra to a symplectic geometry on a manifold. Let M be a real $2n$–dimensional manifold. Then we say that M is an **almost symplectic manifold** if it is equipped with a non–degenerate 2–form Ω. Then (T_zM, Ω_z) is a symplectic vector space for any $z \in M$. If, in addition, Ω is closed (i.e. $d\Omega = 0$), then (M, Ω) is called a **symplectic manifold**. Let (M, Ω) and (M', Ω') be two symplectic manifolds. Then a smooth map $f : M \to M'$ is called a **symplectic morphism** if $f^*\Omega' = \Omega$, that is, at any point $z \in M$ we have

$$\Omega_z(u, v) = \Omega'_{f(z)}(f_*u, f_*v), \quad \forall\, u, v \in T_zM,$$

where f_* is the differential at z of f. It follows that f_* is injective and therefore $\dim M \leq \dim M'$. In particular, if f is a symplectic diffeomorphism then we call it a **symplectomorphism**. The next theorem describes completely the local structure of a symplectic manifold. For its proof we recommend Blair [Bla01], p.8.

Theorem 1.3. (Darboux's Theorem). *Let (M, Ω) be a $2n$–dimensional symplectic manifold and z a point of M. Then there exists a local chart $\{(\mathcal{U}, \varphi) : (x^1, ..., x^n, y^1, ..., y^n)\}$ about z such that Ω is expressed on $\mathcal{U}$ as follows*

$$\Omega = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{1.13}$$

For examples of symplectic manifolds we first consider $\mathbb{R}^{2n}$ with global coordinates $(x^1, ..., x^n, y^1, ..., y^n)$ and Ω expressed as in (1.13). Then $(\mathbb{R}^{2n}, \Omega)$ is a symplectic manifold. The next example has its roots in classical mechanics and it is of great importance in studying symplectic geometry.

Example 1.2. Let T^*M be the cotangent bundle of an n–dimensional manifold M. Let (x^i, y_i), $i \in \{1, ..., n\}$ be the local coordinates on T^*M, where (x^i) are the local coordinates on M and (y_i) are the fiber coordinates. Then the change of coordinates on T^*M is given by

$$\text{(a) } \widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \quad \text{(b) } \widetilde{y}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\, y_j. \tag{1.14}$$

By using (1.14) it is easy to check that

$$\omega = y_i dx^i, \tag{1.15}$$

is a 1–form globally defined on T^*M. This 1–form is known as the **Liouville form** on T^*M. Finally, consider the 2–form

$$\Omega = -d\omega = dx^i \wedge dy_i, \tag{1.16}$$

which is closed and non–degenerate. Thus (T^*M, Ω), where Ω is given by (1.16) is a symplectic manifold. In mechanics, M plays the role of configuration space and T^*M that of phase space (see Sternberg [Ste83], p.144). ■

Now, we want to present an important relationship between symplectic geometry on the one side and Riemannian and complex geometries on the other side. First suppose that (M, J, g) is an almost Kähler manifold with fundamental 2–form Ω given by (2.1.28). As Ω is closed and non–degenerate we conclude that (M, Ω) is a symplectic manifold. The converse is also true (see Blair [Bla01], p. 35). That is to say that any symplectic manifold (M, Ω) admits a Riemannian metric g and an almost complex structure J such that

(M, J, g) is an almost Kähler manifold. We call (g, J) the **associated almost Kähler structure** to the symplectic structure defined by Ω on M. It is important to note that for the symplectic form Ω there is a path of associated metrics g_t, $t \in \mathbb{R}$ (see Blair [Bla01], p.37). Finally, we note that these geometric objects are related by (see (2.1.28))

$$\Omega(X, Y) = g(X, JY), \quad \forall\, X, Y \in \Gamma(TM). \tag{1.17}$$

In conclusion, we can state the following.

Theorem 1.4. *A smooth manifold admits a symplectic structure if and only if it admits an almost Kähler structure.*

From the above theorem it follows that any Kähler manifold admits a symplectic structure. However, the converse is not true. Thurston [Thu76] constructed the first example of compact symplectic manifold that does not admit a Kähler structure.

Next, we consider a submanifold N of a $2n$–dimensional symplectic manifold (M, Ω). Then we use the algebra discussed earlier to classify N according to the behaviour of Ω on the tangent bundle TN. If T_zN is a non–degenerate subspace of (T_zM, Ω_z) for all $z \in N$, then N is called a **symplectic submanifold**. This is because $(N, \Omega_{|TN})$ is also a symplectic manifold. To study the degenerate case we consider the Ω–orthogonal space of T_zN, that is,

$$T_zN_{\Omega}^{\perp} = \{u \in T_zM : \Omega_z(u, v) = 0, \text{ for all } v \in T_zN\}. \tag{1.18}$$

If for every $z \in N$, T_zN is isotropic (coisotropic) subspace of (T_zM, Ω_z) we say that N is an **isotropic** (**coisotropic**) **submanifold** of (M, Ω). Thus N is isotropic (coisotropic) if and only if $T_zN \subset T_zN_{\Omega}^{\perp}$ ($T_zN_{\Omega}^{\perp} \subset T_zN$). In particular, if N is isotropic (coisotropic) then $\dim N \leq n$ ($\dim N \geq n$). If for all $z \in N$, T_zN is a Lagrangian subspace of (T_zM, Ω_z) then we say that N is a **Lagrangian submanifold**. In this case N is necessarily of dimension n. Now, suppose that the radicals $\operatorname{rad} T_zN$, $z \in N$, define an r–distribution on N which we denote by $\operatorname{rad} TN$. In this case, we say that N is a **submanifold of type** r. Then we may describe all the other classes of submanifolds in terms of submanifolds of a certain type as follows.

Theorem 1.5. *Let N be a p–dimensional submanifold of a $2n$–dimensional symplectic manifold (M, Ω). Then we have the following assertions:*

(i) *N is a symplectic submanifold if and only if it is of type $r = 0$.*
(ii) *N is an isotropic submanifold if and only if it is of type $r = p < n$.*
(iii) *N is a coisotropic submanifold if and only if it is of type $r = 2n - p < p$.*
(iv) *N is a Lagrangian submanifold if and only if it is of type $r = p = n$.*

Submanifolds of the types given in the above theorem are abundant. To show this we consider the symplectic manifold (M, Ω) as an almost Kähler manifold (M, J, g). Then a submanifold N of M is an **invariant submanifold** if $J(TN) = TN$. In this case it is easy to see that (N, J, g) is an almost Kähler manifold and therefore $(N, \Omega_{|TN})$ is a symplectic manifold. Thus, all invariant submanifolds are symplectic submanifolds. Hence any complex projective space CP^n is a symplectic submanifold of CP^m, for $n < m$. On the other hand, any curve C in (M, Ω) is an isotropic submanifold because $\Omega_{|TC} = 0$, and therefore C is a submanifold of type $r = 1$. Now, let N be a hypersurface of (M, Ω). Then $T_zN_{\Omega}^{\perp}$ is of dimension 1, and thus $\Omega_z(T_zN_{\Omega}^{\perp}, T_zN_{\Omega}^{\perp}) = 0$ for any $z \in N$. So any hypersurface of a symplectic manifold is coisotropic. Finally, let u be a non–zero vector at $z \in M$. Then, by Lemma 2.1.5 and Theorem 1.3, we may choose the local coordinates (x^i, y^i) about z such that $u = \dfrac{\partial}{\partial y^1}(z)$ and Ω is expressed by (1.13). Thus, $x^i = \text{const.}$, $i \in \{1, ..., n\}$ define a Lagrangian submanifold of (M, Ω) through z and tangent to u.

Here the focus of our attention is on the geometry of Lagrangian submanifolds and Lagrangian foliations. However, as far as we know, the geometry of the other submanifolds of type r from Theorem 1.5 is yet to be settled.

Lagrangian submanifolds play an important role in understanding the local structure of symplectic manifolds. To be more precise, we identify the n-dimensional manifold N with the zero section in T^*N. Thus N can be considered as a Lagrangian submanifold of the symplectic manifold $(T^*N, -d\omega)$ (see Example 1.2). This natural geometric structure turns out to be locally symplectomorphic to any $2n$–dimensional symplectic manifold. The following theorem gives the precise meaning of this equivalence.

Theorem 1.6. (Weinstein [Wei71]). *Let N be a Lagrangian submanifold of a symplectic manifold (M, Ω). Then there exists a neighbourhood of N in M that is symplectomorphic to a neighbourhood of the zero section of T^*N.*

Because Ω vanishes identically on $\Gamma(TN)\times\Gamma(TN)$, there are no geometric objects induced by the symplectic structure on the Lagrangian submanifold N. However, if we consider an associated almost Kähler structure (g, J) to Ω then N inherits an interesting geometric structure as we can see from the next theorem.

Theorem 1.7. *Let N be an n–dimensional submanifold of the $2n$–dimensional symplectic manifold (M, Ω). Then N is a Lagrangian submanifold if and only if it is a totally real submanifold with respect to the associated almost Kähler structure (g, J).*

Proof. N is Lagrangian if and only if $\Omega(X, Y) = 0$ for any $X, Y \in \Gamma(TN)$. Thus, by (1.17), we deduce that N is Lagrangian if and only if

$$g(X, JY) = 0, \quad \forall\, X, Y \in \Gamma(TN). \tag{1.19}$$

According to the terminology we introduced in Example 2.1.8, (1.19) holds if and only if N is totally real. Indeed, (1.19) holds if and only if $J(TN) = TN^{\perp}$, and therefore the holomorphic distribution on N is trivial. ■

Proposition 1.8. *Let N be a Lagrangian submanifold of a symplectic manifold (M, Ω). Then the normal bundle $TN^{\perp}$ with respect to g is a Lagrangian subbundle of $TM_{|N}$, that is, we have*

$$\Omega(U, V) = 0, \quad \forall U, V \in \Gamma(TN^{\perp}).$$

Proof. From the above proof we see that $JV \in \Gamma(TN)$ for any $V \in \Gamma(TN^{\perp})$. Then the assertion follows by using (1.17). ■

If $\{E_i\}$, $i \in \{1, ..., n\}$ is a local orthonormal frame field on N, then $\{E_i, JE_i\}$, $i \in \{1, ..., n\}$ is a local orthonormal frame field on M along N. If $X, Y \in \Gamma(TM_{|N})$, we put

$$X = X^i E_i + X^{n+i} JE_i, \quad Y = Y^i E_i + Y^{n+i} JE_i,$$

and by using (1.17) we obtain

$$\Omega(X, Y) = \sum_{i=1}^{n} \{X^{n+i} Y^i - X^i Y^{n+i}\}.$$

Thus, along a Lagrangian submanifold the symplectic form is expressed as the standard symplectic form of $\mathbb{R}^{2n}$ (see Example 1.1). The above frame field is used in the book Yano–Kon [YK84] to give many results on the geometry of N as a totally real submanifold of (M, g, J).

Next, let (M, Ω) be a $2n$–dimensional symplectic manifold and $\mathcal{F}$ be a foliation on M. Then $\mathcal{F}$ is called a **Lagrange foliation** (cf. Weinstein [Wei71]) if every leaf of $\mathcal{F}$ is a Lagrangian submanifold of M. If $\mathcal{D}$ is the tangent distribution to $\mathcal{F}$, then $\mathcal{F}$ is a Lagrange foliation if and only if the fibers of $\mathcal{D}$ are Lagrangian subspaces of fibers of TM. As a standard example of Lagrange foliation we have the foliation by fibers of the cotangent bundle of a manifold (see Example 1.2). Actually, from the next theorem we see that any Lagrange foliation is locally symplectomorphic to the standard Lagrange foliation of the cotangent bundle. To state this we give the following definition. A Lagrangian submanifold N of (M, Ω) is said to be **transversal** to a Lagrange foliation $\mathcal{F}$ if at any point $x \in N$ we have

$$T_x M = T_x N \oplus \mathcal{D}_x.$$

Now, we can state the following.

Theorem 1.9. (Weinstein [Wei71]). *Let $\mathcal{F}$ be a Lagrange foliation of a symplectic manifold (M, Ω) and N be a transversal Lagrangian submanifold to $\mathcal{F}$. Then there exists a symplectomorphism of a neighbourhood of N in M onto a neighbourhood of the zero section of T^*N which takes the leaves of $\mathcal{F}$ onto the fibers of T^*N.*

From this theorem we deduce the following corollary.

Corollary 1.10. (Weinstein [Wei71]). *Let $\mathcal{F}$ be a Lagrange foliation of the $2n$–dimensional symplectic manifold (M, Ω) and $x \in M$ be any point. Then there exists a symplectomorphism of a neighbourhood of x in M onto a neighbourhood of $0 \in T^*R^n \equiv \mathbb{R}^{2n}$ which takes the leaves of $\mathcal{F}$ onto the fibers of $T^*\mathbb{R}^n$.*

It is interesting to note that the leaves of Lagrange foliations inherit a special geometric structure. More precisely, we have the following.

Theorem 1.11. (Weinstein [Wei71]). *The leaves of a Lagrange foliation on a symplectic manifold are locally affine manifolds. Conversely, if N admits a locally affine structure, then it is a leaf of a Lagrange foliation of some symplectic manifold.*

Remark 1.3. Theorems 1.6, 1.9 and 1.11 and Corollary 1.10 were originally stated by Weinstein [Wei71] in terms of the local manifold pairs modelled on Banach spaces. Here we stated Theorem 1.6 as it is in Blair [Bla01], p. 9, and by using his terminology we stated Theorems 1.9, 1.11 and Corollary 1.10. ■

Now, we are in a position to relate Lagrange foliations on a symplectic manifold with parallel totally–null foliations on semi–Riemannian manifolds (see Sections 4.5 and 4.6).

Theorem 1.12. (Farran [Far79]). *Let $\mathcal{F}$ be a Lagrange foliation on a $2n$–dimensional symplectic manifold (M, Ω). Then M admits a semi–Riemannian metric g such that $\mathcal{F}$ is totally–null and parallel with respect to the Levi–Civita connection on (M, g).*

Proof. From Corollary 1.10 we conclude that M can be covered by the domains of an atlas $\mathcal{A}$, whose transformations of coordinates are local diffeomorphisms of $\mathbb{R}^{2n}$, preserving both the canonical symplectic form of $T^*\mathbb{R}^n$ and its foliation by fibers. Hence $\mathcal{A}$ is a special leaf atlas on M. Thus if (x^i, t^i), $i \in \{1, ..., n\}$, are local coordinates on M, where (x^i) are the leaf coordinates, the change of coordinates is given by (cf. (2.1.5))

$$\widetilde{x}^i = \widetilde{x}^i(x, t), \ \widetilde{t}^i = \widetilde{t}^i(t), \quad i \in \{1, ..., n\}. \tag{1.20}$$

Accordingly, we have

$$\frac{\partial}{\partial x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\,\frac{\partial}{\partial \widetilde{x}^j}\,, \quad \frac{\partial}{\partial t^i} = \frac{\partial \widetilde{x}^j}{\partial t^i}\,\frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{t}^j}{\partial t^i}\,\frac{\partial}{\partial \widetilde{t}^j}\,. \tag{1.21}$$

The canonical symplectic form of $T^*\mathbb{R}^n$ has the matrix (cf. (1.2))

$$[\Omega] = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Taking into account that $[\Omega]$ is preserved by (1.21) we deduce that

$$\frac{\partial \widetilde{x}^i}{\partial x^j} = \frac{\partial t^j}{\partial \widetilde{t}^i}\,.$$

Thus (1.20) becomes

$$\begin{aligned} &\widetilde{x}^i = L^i_j(t)x^j + S^i(t), \ \ L^i_j(t) = \frac{\partial t^j}{\partial \widetilde{t}^i}\,, \\ &\widetilde{t}^i = \widetilde{t}^i(t). \end{aligned} \tag{1.22}$$

Comparing (1.22) with (4.6.14) and using Theorem 4.6.5 we obtain the assertion of the theorem. ■

A converse of the above theorem can be stated as follows.

Theorem 1.13. (Farran [Far79]). *Let $\mathcal{F}$ be a parallel totally–null n–foliation on a $2n$–dimensional semi–Riemannian manifold M. Then M admits an almost symplectic structure such that $\mathcal{F}$ is a Lagrange foliation.*

Proof. Since M is paracompact, there exists a Riemannian metric g on M. On the other hand, by Theorem 4.6.10 there exists on M an almost complex structure J. Now we consider the almost Hermitian metric $\bar{g}$ given by

$$\bar{g}(X,Y) = g(X,Y) + g(JX,JY), \ \ \forall\, X,Y \in \Gamma(M). \tag{1.23}$$

Then it is easy to see that Ω given by

$$\Omega(X,Y) = \bar{g}(X,JY), \ \ \forall\, X,Y \in \Gamma(M), \tag{1.24}$$

is skew–symmetric and non–degenerate. Hence (M,Ω) is an almost symplectic manifold. Next, by using the bundle isomorphism $TM \cong \mathcal{D} \oplus \mathcal{D}$ from Theorem 4.6.10, we can identify any $X \in \Gamma(TM)$ with a pair (U,V), where $U,V \in \Gamma(\mathcal{D})$. In particular, $U \in \Gamma(\mathcal{D})$ can be thought of either as the pair $(U,0)$ or $(0,U)$. Then the almost complex structure on M is given by

$$J(U,V) = (-V,U), \ \ \forall\, U,V \in \Gamma(\mathcal{D}). \tag{1.25}$$

Finally, by using (1.23), (1.24) and (1.25) we obtain

$$\begin{aligned}\Omega(U,V) &= \bar{g}((U,0),J(V,0)) \\ &= g((U,0),(0,V)) - g((0,U),(V,0)) \\ &= 0, \quad \text{for any } U,V \in \Gamma(\mathcal{D}).\end{aligned}$$

Hence any leaf of $\mathcal{F}$ is an isotropic submanifold of (M,Ω). As $\mathcal{F}$ is an n–foliation of a $2n$–dimensional manifold, we conclude that $\mathcal{F}$ is a Lagrange foliation on (M,Ω). This completes the proof of the theorem. ■

We think that the above link between Lagrange foliations and parallel totally–null foliations might be extended to more general foliations on symplectic manifolds.

5.2 Legendre Foliations on Contact Manifolds

Let M be a real $(2n+1)$–dimensional manifold and η be a 1–form on M satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, where the exponent denotes the n^{th} exterior power. Then we say that (M,η) is a **contact manifold** with **contact form** η (cf. Blair [Bla76], p.1). A contact manifold (M,η) admits a natural distribution $\mathcal{H}$. This is simply the subbundle of TM on which $\eta = 0$. To be more specific we write

$$\Gamma(\mathcal{H}) = \{X \in \Gamma(TM) : \eta(X) = 0\}.$$

The distribution $\mathcal{H}$ is called the **contact distribution** on (M,η). Now, we want to relate contact manifolds with the contact metric manifolds defined in Chapter 3 (see Example 3.4.2). We recall that (M,g,φ,ξ,η) is a contact metric manifold, where g is a Riemannian metric, φ is a tensor field of type $(1,1)$, ξ is a vector field and η is a 1–form satisfying:

$$\begin{aligned}&\text{(a) } \varphi^2 = -I + \eta \otimes \xi, \quad \text{(b) } \eta(X) = g(X,\xi), \\ &\text{(c) } g(X,\varphi Y) = d\eta(X,Y), \quad \text{(d) } \eta(\xi) = 1, \quad \text{(e) } \varphi(\xi) = 0, \\ &\text{(f) } \eta(\varphi X) = 0, \quad \text{(g) } g(X,\varphi Y) + g(Y,\varphi X) = 0, \\ &\text{(h) } g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),\end{aligned} \tag{2.1}$$

for any $X,Y \in \Gamma(TM)$. Actually, only (a), (b), (c) have been used to define a contact metric structure, while all the others can be deduced (see (3.4.22), (3.4.23)). It is easy to see that a contact metric manifold is a contact manifold. By the next theorem, the converse is also true (see Blair [Bla76], p.25, Yano–Kon [YK84], p.256).

Theorem 2.1. *Any contact manifold (M,η) admits a contact metric structure (g,φ,ξ,η).*

By (2.1c) we define a 2–form Ω on M by

$$\Omega(X,Y) = g(X,\varphi Y), \quad \forall\, X,Y \in \Gamma(TM), \tag{2.2}$$

and call it the **fundamental** 2–**form** of the contact metric structure (g,φ,ξ,η). It is easy to see that Ω defines a symplectic structure on the contact distribution, that is, Ω is non–degenerate and $d\Omega = 0$ on $\Gamma(\mathcal{H})^3$. The vector field ξ is called the **characteristic vector field** or **Reeb vector field** on the contact manifold (M,η).

Now, we want to examine the integrability of the contact distribution. By (2.1b) we see that the contact distribution $\mathcal{H}$ coincides with the complementary orthogonal distribution to the **characteristic distribution** span$\{\xi\}$. Now, suppose that $\mathcal{H}$ is integrable. Then, for any $X,Y \in \Gamma(\mathcal{H})$ we have $[X,Y] \in \Gamma(\mathcal{H})$, that is $\eta([X,Y]) = 0$. Thus $d\eta(X,Y) = 0$ for any $X,Y \in \Gamma(\mathcal{H})$. As from (2.1c) and (2.1e) we have

$$d\eta(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM), \tag{2.3}$$

we conclude that $d\eta = 0$ on M, which is impossible because M is a contact manifold. Thus we may state the following.

Proposition 2.2. *The contact distribution on a contact manifold is not integrable.*

For the exterior derivative of η we use the formula (cf. Kobayashi–Nomizu [KN63], p.36)

$$d\eta(X,Y) = \frac{1}{2}\left(X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\right), \tag{2.4}$$

for any $X,Y \in \Gamma(TM)$. Then, by using (2.3), (2.4), (2.1d) and (2.1b) we obtain

$$\eta([X,\xi]) = 0, \quad \forall\, X \in \Gamma(\mathcal{H}). \tag{2.5}$$

Now, suppose that N is a p–dimensional integral manifold of the contact distribution $\mathcal{H}$. Then, by (2.4), we obtain

$$d\eta(X,Y) = 0, \quad \forall\, X,Y \in \Gamma(TN). \tag{2.6}$$

Hence, by (2.1c), we deduce that $g(X,\varphi Y) = 0$, which means that $\varphi(TN) \subset TN^{\perp}$. Therefore, N is an anti–invariant submanifold of (M,g,φ,ξ,η), which is normal to ξ (cf. Yano–Kon [YK84], p.344). As φ is an automorphism of $\Gamma(\mathcal{H})$ we conclude that $p < n+1$. Hence the maximum dimension of an integral manifold of $\mathcal{H}$ is $p = n$. Fortunately, there exist integral manifolds of maximum dimension. To show this we present the following theorem (see a proof in Blair [Bla01], p.18).

Theorem 2.3. (Darboux's Theorem). *Let (M,η) be a $(2n+1)$–dimensional contact manifold. Then about each point there exists local coordinates $(x^1,...,x^n,y^1,...,y^n,z)$ such that*

$$\eta = dz - \sum_{i=1}^{n} y^i dx^i. \tag{2.7}$$

Then from (2.7) it follows that $x^i = \text{const.}, z = \text{const.}$, $i \in \{1,...,n\}$, define an n–dimensional integral manifold of $\mathcal{H}$.

Summing up the above discussion, we may state the following.

Theorem 2.4. *Let (M,η) be a $(2n+1)$–dimensional contact manifold. Then there exist integral manifolds of the contact distribution $\mathcal{H}$ of dimension n, but of no higher dimension.*

We now present some examples of contact manifolds.

Example 2.1. (Blair [Bla76], p.7). Consider (x^i,y^i,z), $i \in \{1,...,n\}$, as Cartesian coordinates on $\mathbb{R}^{2n+1}$ and define the 1–form $\eta = dz - \sum_{i=1}^{n} y^i dx^i$. Then $(\mathbb{R}^{2n+1},\eta)$ is a contact manifold with contact distribution $\mathcal{D}$ spanned by

$$X_i = \frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}\,, \;\; X_{n+i} = \frac{\partial}{\partial y^i}\,, \;\; i \in \{1,...,n\},$$

and with characteristic vector field $\xi = \frac{\partial}{\partial z}\cdot$ ∎

Example 2.2. Let T^*M be the cotangent bundle of an $(n+1)$–dimensional manifold. We take (x^i,y_i), $i \in \{1,...,n+1\}$ as local coordinates on T^*M, where (x^i) are the local coordinates on M and (y_i) are the fiber coordinates. Now, we consider the open submanifold $T^*_\circ M$ of non–zero covectors in T^*M, and suppose there exists a function $F : T^*M \longrightarrow [0,\infty)$ that is smooth on $T^*_\circ M$ and satisfies

$$F(tv) = tF(v), \quad \text{for all } t \geq 0 \;\text{ and }\; v \in T^*M.$$

Then $S^*_F M = \{v \in T^*M : F(v) = 1\}$ is a hypersurface of T^*M, and therefore is a $(2n+1)$–dimensional manifold. If we consider the Liouville form $\omega = y_i dx^i$ on T^*M (see Example 1.2), then $(S^*_F M,\eta)$ is a contact manifold, where η is the pull–back to $S^*_F(M)$ of ω. This example is due to Pang [Pan90]. In particular, if g is a Riemannian metric on M, then we can consider F as the norm defined by g. In this case $S^*_F M$ is known as the cotangent sphere bundle of the Riemannian manifold(M,g) (cf. Blair [Bla01], p.22). In general, we call $S^*_F M$ the **unit cotangent bundle** with respect to F. ∎

Next, following Pang [Pan90] we give the following definition. A **Legendre foliation** of a contact manifold (M, η) is a foliation of M by n–dimensional integral manifolds of the contact distribution $\mathcal{H}$. Thus a foliation $\mathcal{F}$ of (M, η) is a Legendre foliation if and only if the distribution $\mathcal{D}$ tangent to $\mathcal{F}$ is an n-subbundle of the $2n$–distribution $\mathcal{H}$. The main results on the geometry of Legendre foliations can be found in Pang [Pan90], Libermann [Lib91] and Jayne [Jay92], [Jay94]. Some of these results will be presented in the remaining part of this section.

Now, we give two examples of Legendre foliations. If $(\mathbb{R}^{2n+1}, \eta)$ is the contact manifold in Example 2.1, then the two distributions spanned by $\{X_i\}$ and $\{X_{n+i}\}$, $i \in \{1, ..., n\}$, are integrable. Thus there exist two complementary Legendre foliations on $(\mathbb{R}^{2n+1}, \eta)$. Next, we consider the unit cotangent bundle $S_F^* M$ from Example 2.2. Then the foliation by fibers of the projection map $\pi : S_F^* M \longrightarrow M$ is a Legendre foliation, which we denote by $\mathcal{F}_F$.

Next, let (M, η) and $(\widetilde{M}, \widetilde{\eta})$ be two contact manifolds of the same dimension $2n+1$. Then two Legendre foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ on M and $\widetilde{M}$ respectively, are said to be **locally equivalent** if for any point $x \in M$ there exist a neighbourhood $\mathcal{U}$ of x and a dioffeomorphism $\Phi : \mathcal{U} \longrightarrow \widetilde{\mathcal{U}}$, where $\widetilde{\mathcal{U}}$ is a neighbourhood of $\Phi(x)$, such that

$$\Phi^* \widetilde{\eta}_{|\widetilde{\mathcal{U}}} = \eta_{|\mathcal{U}} \quad \text{and} \quad \Phi^{-1}(\widetilde{\mathcal{F}}_{|\widetilde{\mathcal{U}}}) = \mathcal{F}_{|\mathcal{U}},$$

where $\Phi^* \widetilde{\eta}_{\widetilde{\mathcal{U}}}$ is the pull–back to $\mathcal{U}$ of $\widetilde{\eta}_{|\widetilde{\mathcal{U}}}$ and $\Phi^{-1}(\widetilde{\mathcal{F}}_{|\widetilde{\mathcal{U}}})$ is the foliation of $\mathcal{U}$ whose leaves are the inverse images under Φ of leaves of $\widetilde{\mathcal{F}}_{\widetilde{\mathcal{U}}}$. Now, we state the following theorem about local equivalence of Legendre foliations.

Theorem 2.5. (Pang [Pan90]). *Any Legendre foliation $\mathcal{F}$ is locally equivalent with one of the form $\mathcal{F}_F$.*

Moreover, Pang [Pan90] shows that the above theorem generalizes to a global equivalence theorem, provided the leaves of $\mathcal{F}$ are compact and simply connected. It is interesting to note that in this case F defines a Finsler metric on the manifold.

Following some ideas from Finsler geometry Pang [Pan90] defined two invariants on a Legendre foliation $\mathcal{F}$ on (M, η). To present them we denote by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$. Then the first invariant is a symmetric $F(M)$–bilinear form Π on $\Gamma(\mathcal{D})$ given by

$$\Pi(X, Y) = -(\mathcal{L}_X \mathcal{L}_Y \eta)(\xi), \quad \forall\, X, Y \in \Gamma(\mathcal{D}), \tag{2.8}$$

where $\mathcal{L}$ is the Lie derivative on M. By elementary calculations, using (2.1d) and (2.5) we obtain

$$\Pi(X, Y) = \eta([Y, [\xi, X]]). \tag{2.9}$$

We remark that Π does not depend on either, the Riemannian metric g or the tensor field φ of any contact metric structure (g, φ, ξ, η). However, by using (2.9), (2.4), (2.5), (2.1e) and (2.1g) we deduce that (cf. Jayne [Jay92], p.32)

$$\Pi(X,Y) = 2g([\xi, X], \varphi Y). \tag{2.10}$$

The Legendre foliation $\mathcal{F}$ is called **flat** when Π vanishes identically on M. Two interesting characterizations for this class of foliations are given in the next theorem.

Theorem 2.6. *Let $\mathcal{F}$ be a Legendre foliation on the contact manifold (M, η). Then the following assertions are equivalent:*

(i) *$\mathcal{F}$ is flat.*
(ii) *$[\xi, X] \in \Gamma(\mathcal{D})$, for any $X \in \Gamma(\mathcal{D})$.*
(iii) *$\mathcal{F}$ is invariant with respect to the actions of all local flows of ξ.*

Proof. The equivalence of (i) and (ii) is obtained by using (2.5), (2.10), and by taking into account that the leaves of $\mathcal{D}$ are anti–invariant submanifolds. Finally, by Lemma 2.3.5 we deduce the equivalence of (ii) and (iii). ∎

Also, some results of Weinstein (see Theorem 1.11 and Corollary 1.10) for Lagrange foliations have been extended to flat Legendre foliations.

Theorem 2.7. (Pang [Pan90]). *Let $\mathcal{F}$ be a flat Legendre foliation on (M, η) and $x \in M$. Then there are coordinates (x^i, y^i, z) about x, such that*

$$\eta = dz + \sum_{i=1}^{n} y^i dx^i,$$

and the foliation is defined by $x^i =$ const. *and* $z =$ const. *Moreover, the leaves of $\mathcal{F}$ are locally affine manifolds.*

Jayne [Jay92], p. 63, has presented an example of a metric manifold which admits five different contact metric structures. Corresponding to each contact metric structure he defined a flat Legendre foliation. Four of these foliations are totally geodesic and one is harmonic.

When Π is non–degenerate (resp. positive definite) on $\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D})$, the Legendre foliation is called **non–degenerate** (resp. **positive definite**). The theory of non–degenerate Legendre foliations was developed by Pang [Pan90] as a generalization of Finsler manifolds. More precisely, he extended Chern's theory on Finsler manifolds (Chern [Che48]) to non–degenerate Legendre foliations. In particular, he proved the following.

Theorem 2.8. (Pang [Pan90]). *A Legendre foliation $\mathcal{F}$ is locally equivalent to one of the form $\mathcal{F}_F$ with F a Finsler metric, if and only if it is positive definite.*

The second invariant for a Legendre foliation was also introduced by Pang [Pan90] as follows:

$$G(X,Y,Z) = \frac{1}{2}\{X\Pi(Y,Z) + Y\Pi(Z,X) + Z\Pi(X,Y) + (\mathcal{L}_Y\mathcal{L}_X\mathcal{L}_Z\eta + \mathcal{L}_Z\mathcal{L}_X\mathcal{L}_Y\eta)\xi\},$$

for any $X, Y, Z \in \Gamma(\mathcal{D})$.

Theorem 2.9. (Pang [Pan90]). *A non–degenerate Legendre foliation $\mathcal{F}$ is locally equivalent to $\mathcal{F}_F$ with F a norm defined by a semi–Riemannian metric if and only if $G = 0$. When non–degeneracy is replaced by positive definiteness, then the semi–Riemannian metric is Riemannian.*

The next theorem states the local structure of any contact manifold that admits a Legendre foliation.

Theorem 2.10. (Libermann [Lib91]). *Let $\mathcal{F}$ be a Legendre foliation on a contact manifold (M,η). Then for any $x \in M$, there exists an open neighbourhood $\mathcal{U}$ which admits local coordinates $(x^1, ..., x^n, p_1, ..., p_n, t)$ such that η is given by*

$$\eta = \sum_{i=1}^{n} p_i dx^i - H dt,$$

with H a function of (x^i, p_i, t) satisfying the condition: the function

$$A = \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} - H$$

has no zero. By means of these coordinates the characteristic vector field ξ is expressed by

$$\xi = \frac{1}{A}\left(\frac{\partial}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}\right)\right),$$

and the symmetric bilinear form Π on $\mathcal{D}$ is given by

$$\Pi\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = -\frac{1}{A}\frac{\partial^2 H}{\partial p_i \partial p_j}.$$

Next, we suppose that (g, φ, ξ, η) is a contact metric structure on the contact manifold (M,η). As M carries the Riemannian metric g, it is interesting to study the conditions for a Legendre foliation to fall into one of the classes of foliations presented in Chapter 3. First we fix some notations. If $\mathcal{D}$ is the tangent distribution to the Legendre foliation $\mathcal{F}$, then $\mathcal{D}^\perp$ represents the complementary orthogonal distribution to $\mathcal{D}$ in TM with respect to g. As any integral manifold of $\mathcal{D}$ is anti–invariant with respect to φ, we have

$$g(X, \varphi Y) = 0, \quad \forall\, X, Y \in \Gamma(\mathcal{D}). \tag{2.11}$$

Hence the contact distribution $\mathcal{H}$ has the following decomposition

$$\mathcal{H} = \mathcal{D} \oplus \varphi\mathcal{D}, \tag{2.12}$$

where

$$\varphi\mathcal{D} = \{\varphi X : \forall\, X \in \Gamma(\mathcal{D})\}.$$

Now, according to Theorem 3.3.1, we deduce that g is bundle–like for $\mathcal{F}$ if and only if

$$X(g(U,V)) = g([X,U],V) + g([X,V],U), \tag{2.13}$$

for any $X \in \Gamma(\mathcal{D})$ and $U, V \in \Gamma(\mathcal{D}^{\perp})$.

Lemma 2.11. *Let $\mathcal{F}$ be a Legendre foliation on the contact metric manifold $(M, g, \varphi, \xi, \eta)$ such that*

$$g([X,\varphi Y],\xi) + g([X,\xi],\varphi Y) = 0, \quad \forall\, X, Y \in \Gamma(\mathcal{D}). \tag{2.14}$$

Then we have

$$\Pi(X,Y) = 4g(X,Y), \quad \forall\, X, Y \in \Gamma(\mathcal{D}). \tag{2.15}$$

Proof. By using (2.10) and (2.1b), (2.14) becomes

$$\eta([X,\varphi Y]) - \frac{1}{2}\, \Pi(X,Y) = 0. \tag{2.16}$$

Finally, by using (2.4), (2.1c) and (2.1a) in (2.16) we obtain (2.15). ■

Theorem 2.12. *Let $\mathcal{F}$ be a Legendre foliation on $(M, g, \varphi, \xi, \eta)$ such that g is bundle–like Riemannian metric for $\mathcal{F}$. Then $\mathcal{F}$ is locally equivalent to one of the form $\mathcal{F}_F$ with F a Finsler metric.*

Proof. We replace U and V from (2.13) by φY and ξ respectively and obtain (2.14). Then, from (2.15) we deduce that $\mathcal{F}$ is positive definite. Finally, the assertion follows by applying Theorem 2.8. ■

Now, we consider the second fundamental form h (see 3.2.5) of a Legendre foliation $\mathcal{F}$ on $(M, g, \varphi, \xi, \eta)$. Also, denote by H the mean curvature vector field of $\mathcal{F}$. Finally, we recall (see Section 3.2) that the Levi–Civita connection $\widetilde{\nabla}$ induces two linear connections ∇ and $\nabla^{\perp}$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. Then we say that $\mathcal{F}$ has **parallel second fundamental form** if we have

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0, \tag{2.17}$$

for any $X \in \Gamma(TM)$ and $Y, Z \in \Gamma(\mathcal{D})$. Similarly, we say that the mean curvature vector H of $\mathcal{F}$ is **parallel** if we have

$$\nabla^{\perp}_X H = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.18}$$

If (2.17) and (2.18) are satisfied only for $X \in \Gamma(\mathcal{D})$, then we say that h and H are $\mathcal{D}$–**parallel**. Clearly, when h and H are parallel they are $\mathcal{D}$–parallel. Surprisingly, the converse is also true, provided $(M, g, \varphi, \xi, \eta)$ is a K–contact manifold (see the assertions (i) and (ii) in Theorem 2.14). We say that $(M, g, \varphi, \xi, \eta)$ is a K–**contact manifold** if ξ is a Killing vector field. In this case, we have (cf. Blair [Bla76], p.64)

$$\widetilde{\nabla}_X \xi = -\varphi X, \quad \forall\, X \in \Gamma(TM). \tag{2.19}$$

First, we need the following lemma.

Lemma 2.13. *Let $\mathcal{F}$ be a Legendre foliation on a K–contact manifold $(M, g, \varphi, \xi, \eta)$. Then we have*

$$g(h(X,Y), \xi) = 0, \quad \forall\, X, Y \in \Gamma(\mathcal{D}). \tag{2.20}$$

Proof. Taking into account that $\mathcal{D}$ is anti–invariant with respect to φ and by using (2.19), (1.5.9) and (3.2.8a) we obtain

$$\begin{aligned} 0 = g(X, \varphi Y) &= -g(X, \widetilde{\nabla}_Y \xi) \\ &= g(\widetilde{\nabla}_Y X, \xi) = g(h(X,Y), \xi), \end{aligned}$$

for any $X, Y \in \Gamma(\mathcal{D})$. ■

Now, we can prove the following.

Theorem 2.14. (Jayne [Jay92]). *Let $\mathcal{F}$ be a Legendre foliation on a $(2n+1)$–dimensional K–contact manifold $(M, g, \varphi, \xi, \eta)$. Then we have the following assertions:*

(i) *If the second fundamental form of $\mathcal{F}$ is $\mathcal{D}$–parallel, then $\mathcal{F}$ is a totally geodesic foliation.*
(ii) *If the mean curvature vector of $\mathcal{F}$ is $\mathcal{D}$–parallel, then $\mathcal{F}$ is a harmonic foliation.*
(iii) *If $(M, g, \varphi, \xi, \eta)$ is a $(2n+1)$–dimensional Sasakian manifold with $n > 1$ and $\mathcal{F}$ is totally umbilical, then it is totally geodesic.*

Proof. First, we suppose (2.17) is satisfied for any $X, Y, Z \in \Gamma(\mathcal{D})$. Then, by using (1.5.2), (2.20) and (2.19) we deduce that

$$\begin{aligned} 0 = g((\overline{\nabla}_X h)(Y,Z), \xi) &= -g(h(Y,Z), \widetilde{\nabla}_X \xi) \\ &= g(h(Y,Z), \varphi X), \quad \forall\, X, Y, Z \in \Gamma(\mathcal{D}). \end{aligned} \tag{2.21}$$

Thus, the assertion (i) follows from (2.21) and (2.20) since by (2.12) $\mathcal{D}^{\perp} = \varphi\mathcal{D} \oplus \text{span}\{\xi\}$. Now, suppose H is $\mathcal{D}$–parallel. Then, by using (2.18), (3.2.8b), (1.5.9) and (2.19) we obtain

$$\begin{aligned} 0 = g(\nabla^{\perp}_X H, \xi) = g(\widetilde{\nabla}_X H, \xi) &= -g(H, \widetilde{\nabla}_X \xi) \\ &= g(H, \varphi X), \ \forall X \in \Gamma(\mathcal{D}). \end{aligned} \tag{2.22}$$

On the other hand, by (2.20) and (3.4.28) we infer that

$$g(H, \xi) = 0. \tag{2.23}$$

Then, from (2.22) and (2.23) we deduce the assertion (ii). Finally, we suppose that $\mathcal{F}$ is totally umbilical that is, we have (cf. (3.4.39))

$$h(X, Y) = g(X, Y)H, \quad \forall X, Y \in \Gamma(\mathcal{D}). \tag{2.24}$$

Next, we consider an orthonormal frame field $\{E_i\}$, $i \in \{1, ..., n\}$, for the tangent distribution $\mathcal{D}$. Then, by using (2.24) and (3.2.8a) we obtain

$$\begin{aligned} g(H, \varphi E_i) &= g(g(E_j, E_j)H, \varphi E_i) \\ &= g(h(E_j, E_j), \varphi E_i) \\ &= g(\widetilde{\nabla}_{E_j} E_j, \varphi E_i), \end{aligned} \tag{2.25}$$

for any $i, j \in \{1, ..., n\}$. On the other hand, by using (1.5.9), (3.4.24), (2.1g), (3.2.8a) and (2.24) we deduce that

$$\begin{aligned} g(\widetilde{\nabla}_{E_j} E_j, \varphi E_i) &= -g(E_j, \widetilde{\nabla}_{E_j} \varphi E_i) \\ &= -g(E_j, \varphi \widetilde{\nabla}_{E_j} E_i) \\ &= g(\varphi E_j, \widetilde{\nabla}_{E_j} E_i) \\ &= g(\varphi E_j, h(E_j, E_i)) \\ &= g(\varphi E_j, g(E_j, E_i)H) = 0, \ \text{for } i \neq j. \end{aligned} \tag{2.26}$$

As $n > 1$, from (2.25) and (2.26) we conclude that

$$g(H, \varphi E_i) = 0, \quad \forall i \in \{1, ..., n\}. \tag{2.27}$$

Thus (2.27) and (2.23) (which is true for any Legendre foliation on a K–contact manifold) imply $H = 0$. Hence by (2.24) we obtain $h = 0$, that is, $\mathcal{F}$ is totally geodesic. ∎

Jayne [Jay92] also studied an interesting class of Legendre foliations. To present it, let us first consider a Legendre foliation $\mathcal{F}$ on the contact metric manifold $(M, g, \varphi, \xi, \eta)$ with tangent distribution $\mathcal{D}$. When the distribution $\varphi\mathcal{D}$ (which is orthonormal to $\mathcal{D}$) is integrable, it is said that the foliation $\overline{\mathcal{F}}$

determined by $\varphi\mathcal{D}$ is the **conjugate foliation** to $\mathcal{F}$. It is easy to see that when M is a 3–dimensional contact metric manifold the conjugate foliation exists on M. Indeed, in this case, the contact distribution is of rank 2 and hence the existence of $\mathcal{F}$ implies the existence of $\overline{\mathcal{F}}$, since $\varphi\mathcal{D}$ is a line distribution. Also, when $\mathcal{F}$ is a flat Legendre foliation on (M, η), Jayne [Jay92] constructed a canonical contact metric structure on M with respect to which the conjugate foliation $\overline{\mathcal{F}}$ exists and is flat too.

To state another interesting result on the existence of conjugate foliations we first define some geometric objects on a contact metric manifold $(M, g, \varphi, \xi, \eta)$. Let ψ be a tensor field of type $(1, 1)$ on M given by

$$\psi X = \frac{1}{2}\,(\mathcal{L}_\xi \varphi)X, \quad \forall\, X \in \Gamma(TM). \tag{2.28}$$

Among the properties of ψ we only need the following

$$\text{(a) } \psi\xi = 0 \quad \text{and} \quad \text{(b) } \psi\varphi + \varphi\psi = 0. \tag{2.29}$$

Remark 2.3. In most of the papers published on the geometry of contact metric structures we find the above tensor field denoted by h (cf. Blair [Bla76], p.66). We changed this notation because throughout this book, h denotes the second fundamental form of a foliation. ∎

According to Blair et al. [BKP95] the (k, μ)–**nullity distribution** of a contact metric manifold $(M, g, \varphi, \xi, \eta)$ for the pair $(k, \mu) \in \mathbb{R}^2$ is the distribution

$$\begin{aligned} &N(k,\mu) : x \longrightarrow N_x(k,\mu), \qquad \text{where} \\ &N_x(k,\mu) = \{Z \in T_xM : R(X,Y)Z = k(g(Y,Z)X \\ &\qquad -g(X,Z)Y) + \mu(g(Y,Z)\psi X - g(X,Z)\psi Y)\}. \end{aligned}$$

Now we can state the following.

Theorem 2.15. (Blair–Koufogiorgos–Papantoniou [BKP95]). *Let $(M, g, \varphi, \xi, \eta)$ be a contact metric manifold with ξ belonging to the (k, μ)–nullity distribution. Then $k \leq 1$. If $k = 1$, then $\psi = 0$ and M is a Sasakian manifold. If $k < 1$, then M admits three mutually orthogonal and integrable distributions $\mathcal{D}(0), \mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ determined by the eigenspaces of ψ, where $\lambda = \sqrt{1-k}$.*

Also, the authors of the above paper proved that the tensor fields φ and ψ are related by

$$\psi^2 = (k-1)\varphi^2. \tag{2.30}$$

Since φ is an almost complex structure on the contact distribution $\mathcal{H}$, from (2.30) we deduce that

$$\psi^2 X = (1-k)X, \quad \forall\, X \in \Gamma(\mathcal{H}).$$

Thus two of the eigenvalues of ψ are $\sqrt{1-k}$ and $-\sqrt{1-k}$. Moreover, from (2.29b) it follows that if X is eigenvector for $\sqrt{1-k}$, then φX is eigenvector for $-\sqrt{1-k}$. Hence, the distributions $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are both n–distributions on M. Finally, from (2.29a) we infer that $\mathcal{D}(0)$ is spanned by ξ. Thus, from Theorem 2.15 we obtain the following corollary.

Corollary 2.16. *Let $(M, g, \varphi, \xi, \eta)$ be a contact metric manifold with ξ belonging to the (k, μ)–nullity distribution. Then the Legendre foliations $\mathcal{F}(\lambda)$ and $\mathcal{F}(-\lambda)$ whose tangent distributions are $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ respectively, are conjugate to each other.*

So far, we presented results about flat or non–degenerate Legendre foliations whose symmetric bilinear form Π is vanishing or has maximum rank n respectively. If the rank of Π is between 1 and $n-1$ we say that the Legendre foliation is *degenerate*. Very little is known about this type of Legendre foliations. If $\mathcal{D}$ is the tangent distribution to a degenerate foliation $\mathcal{F}$ it was proved by Libermann [Lib91] and Pang [Pan90] that the totally null distribution $\mathcal{N}$ is integrable and its leaves are locally affine manifolds. This result might have some connections with the general theory of parallel partially–null foliations (see Section 4.7).

5.3 Foliations on the Tangent Bundle of a Finsler Manifold

Let M be a real m–dimensional manifold and TM the tangent bundle of M with canonical projection $\pi : TM \to M$. Then a local chart $(\mathcal{U}, \varphi)$ on M with local coordinates (x^a) for $x \in \mathcal{U}$, $a \in \{1, ..., m\}$, defines a local chart $(\pi^{-1}(\mathcal{U}), \Phi)$ on TM with local coordinates (x^a, y^a) for $y = y^a \left.\dfrac{\partial}{\partial x^a}\right|_x \in \pi^{-1}(\mathcal{U})$. The coordinate transformations on TM are given by

$$\widetilde{x}^a = \widetilde{x}^a(x^1, ..., x^m), \quad \widetilde{y}^a = J^a_b(x) y^b, \tag{3.1}$$

where $J^a_b(x) = \dfrac{\partial \widetilde{x}^a}{\partial x^b} \cdot$ As a consequence of (3.1) the local frame fields $\left\{\dfrac{\partial}{\partial x^a}, \dfrac{\partial}{\partial y^a}\right\}$ and $\left\{\dfrac{\partial}{\partial \widetilde{x}^a}, \dfrac{\partial}{\partial \widetilde{y}^a}\right\}$ are related by

$$\frac{\partial}{\partial x^a} = J^b_a(x) \frac{\partial}{\partial \widetilde{x}^b} + J^b_{ac}(x) y^c \frac{\partial}{\partial \widetilde{y}^b}, \quad J^b_{ac}(x) = \frac{\partial^2 \widetilde{x}^b}{\partial x^a \partial x^c}, \tag{3.2}$$

and

$$\frac{\partial}{\partial y^a} = J^b_a(x) \frac{\partial}{\partial \widetilde{y}^b} \cdot \tag{3.3}$$

Denote by θ the zero section of TM and consider $TM^\circ = TM\backslash\theta(M)$. Suppose that there exists a function $F : TM \longrightarrow [0,\infty)$ which vanishes only on the zero section of TM and is smooth on TM°. Moreover, for any local chart $(\pi^{-1}(\mathcal{U}), \Phi; x^a, y^a)$ on TM°, F satisfies the following conditions:

(F_1) It is positively homogeneous of degree one with respect to (y^a), i.e., we have

$$F(x^a, ky^a) = kF(x^a, y^a), \quad a \in \{1, ..., m\},$$

for any $k > 0$.

(F_2) The matrix

$$[g_{bc}(x^a, y^a)] = \left[\frac{1}{2}\ \frac{\partial^2 F^2}{\partial y^b \partial y^c}\right], \quad a, b, c \in \{1, ..., m\}, \tag{3.4}$$

is positive definite on $\Phi(\pi^{-1}(\mathcal{U}))$. Then we say that $\mathbb{F}^m = (M, F)$ with F satisfying (F_1) and (F_2) is a **Finsler manifold** and F is the **fundamental function** of $\mathbb{F}^m$.

Remark 3.1. The fundamental function F of $\mathbb{F}^m$ is surjective on $\mathbb{R}_+ = (0,\infty)$. Indeed, let $(x, y) \in TM^\circ$ with $y \neq 0$ such that $F(x, y) = a$. Then by the homogeneity of F we deduce that $F\left(x, \frac{c}{a}\ y\right) = c$ for any $c \in \mathbb{R}_+$. ■

A more general concept of Finsler manifold has been considered by the authors in [BF00a], wherein F is smooth on an open submanifold of TM°. Moreover, the condition (F_2) is replaced by

(F_2') $[g_{bc}(x^a, y^a)]$ is non–degenerate of constant index.

However, here we consider the above classical concept of Finsler manifold which enables us to emphasize the role of foliations in Finsler geometry.

Clearly, any Riemannian manifold (M, g) is an example of Finsler manifold. Indeed, the fundamental function is

$$F(x^a, y^a) = (g_{bc}(x^a)y^b y^c)^{1/2},$$

where $g_{bc}(x^a)$ are the local components of g. Now, suppose that (M, g) is endowed with a 1–form η such that $\|\eta\| < 1$, where the norm is considered with respect to g. Then

$$F(x^a, y^a) = (g_{bc}(x^a)y^b y^c)^{\frac{1}{2}} + \eta_a(x)y^a,$$

is a positive function on TM° that satisfies (F_1) and (F_2). The Finsler manifold with the above fundamental function is known as **Randers manifold**. The classification of an important class of Randers manifolds of positive constant curvature has been recently obtained by the authors [BF02], [BF03c]. More examples of Finsler (pseudo–Finsler) manifolds can be found in Bejancu–Farran [BF00a] and Matsumoto [Mat86].

We show here that the geometry of a Finsler manifold $\mathbb{F}^m = (M, F)$ is strongly related to the geometry of some foliations on TM°. First, we recall that the vertical bundle VTM° of TM° is the tangent distribution to the foliation determined by fibers of $\pi : TM^\circ \longrightarrow M$ (see Section 2.1). If (x^a, y^a) are local coordinates on TM°, then VTM° is locally spanned by $\left\{\dfrac{\partial}{\partial y^a}\right\}$, $a \in \{1, ..., m\}$. In this case a **canonical transversal distribution** is constructed as follows (cf. Bejancu–Farran [BF00a], p.38). Denote by $[g^{ab}(x, y)]$ the inverse matrix of $[g_{ab}(x, y)]$ from (3.4). Then locally define the functions

$$G^a(x, y) = \frac{1}{4}\, g^{ab}(x, y) \left(\frac{\partial^2 F^2}{\partial y^b \partial x^c}\, y^c - \frac{\partial F^2}{\partial x^b} \right)(x, y). \tag{3.5}$$

Then there exists on TM° an m–distribution HTM° locally spanned by the vector fields

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - G_a^b(x, y)\, \frac{\partial}{\partial y^b}\,, \quad a \in \{1, ..., m\}, \tag{3.6}$$

where $G_a^b(x, y)$ are given by

$$G_a^b(x, y) = \frac{\partial G^b}{\partial y^a}\,. \tag{3.7}$$

Moreover, it is easily seen that HTM° is complementary to VTM° in TTM°. By using the decomposition

$$TTM^\circ = HTM^\circ \oplus VTM^\circ, \tag{3.8}$$

we define the Riemannian metric G on TM° by the matrix

$$G_{AB}(x, y) = \begin{bmatrix} g_{ab}(x, y) & 0 \\ 0 & g_{ab}(x, y) \end{bmatrix}, \quad \begin{array}{l} A, B \in \{1, ..., 2m\}, \\ a, b \in \{1, ..., m\}. \end{array} \tag{3.9}$$

This means that with respect to the semi–holonomic frame field $\left\{\dfrac{\delta}{\delta x^a}, \dfrac{\partial}{\partial y^a}\right\}$ locally defined on TM°, we have

$$\begin{aligned} &\text{(a)}\quad G\left(\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right) = G\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) = g_{ab}, \\ &\text{(b)}\quad G\left(\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^b}\right) = 0. \end{aligned} \tag{3.10}$$

Thus the two distributions VTM° and HTM° are complementary orthogonal with respect to G. The Riemannian metric G is known as the **Sasaki–Finsler metric** on TM°.

The above discussion shows us that on the Riemannian manifold (TM°, G) we have a foliation $\mathcal{F}_V$ with VTM° and HTM° as structural and transversal distributions respectively. Thus, the theory we developed so far for non–degenerate foliations on semi–Riemannian manifolds applies to the present foliation $\mathcal{F}_V$, which from now on is called the **vertical foliation on** (TM, G)**.**

First, from (2.3.21), (2.2.18) and (2.2.19) we deduce that

$$\begin{aligned} &\text{(a)} \quad \left[\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^b}\right] = \frac{\partial G_a^c}{\partial y^b}\, \frac{\partial}{\partial y^c}\,, \\ &\text{(b)} \quad \left[\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right] = R^c{}_{ab}\, \frac{\partial}{\partial y^c}\,, \end{aligned} \tag{3.11}$$

where we set

$$R^c{}_{ab} = \frac{\delta G_a^c}{\delta x^b} - \frac{\delta G_b^c}{\delta x^a}\,. \tag{3.12}$$

Then denote by $\widetilde{\nabla}$ the Levi–Civita connection on (TM°, G) and by ∇^* and ∇° the Vrănceanu and Schouten–Van Kampen connections defined by $\widetilde{\nabla}$ (see Sections 3.1 and 3.2). If D and $D^\perp$ are the intrinsic linear connections on VTM° and HTM° respectively (see (3.1.10)), then, according to (3.1.22) and (3.1.23), we put

$$\begin{aligned} &\text{(a)} \ \nabla^*_{\frac{\partial}{\partial y^b}} \frac{\partial}{\partial y^a} = D_{\frac{\partial}{\partial y^b}} \frac{\partial}{\partial y^a} = C_a{}^c{}_b \frac{\partial}{\partial y^c}\,, \\ &\text{(b)} \ \nabla^*_{\frac{\delta}{\delta x^b}} \frac{\partial}{\partial y^a} = D_{\frac{\delta}{\delta x^b}} \frac{\partial}{\partial y^a} = G_a{}^c{}_b \frac{\partial}{\partial y^c}\,, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\text{(a)} \ \nabla^*_{\frac{\partial}{\partial y^b}} \frac{\delta}{\delta x^a} = D^\perp_{\frac{\partial}{\partial y^b}} \frac{\delta}{\delta x^a} = L_a{}^c{}_b \frac{\delta}{\delta x^c}\,, \\ &\text{(b)} \ \nabla^*_{\frac{\delta}{\delta x^b}} \frac{\delta}{\delta x^a} = D^\perp_{\frac{\delta}{\delta x^b}} \frac{\delta}{\delta x^a} = F_a{}^c{}_b \frac{\delta}{\delta x^c}\,. \end{aligned} \tag{3.14}$$

Then we state the following.

Proposition 3.1. *The local coefficients of the intrinsic connections D and $D^\perp$ on VTM° and HTM° with respect to the semi–holonomic frame field $\left\{\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a}\right\}$ are given by*

$$\begin{aligned} &\text{(a)} \quad C_a{}^c{}_b = \frac{1}{2}\, g^{cd}\, \frac{\partial g_{ab}}{\partial y^d}\,, \\ &\text{(b)} \quad G_a{}^c{}_b = \frac{\partial G_b^c}{\partial y^a}\,, \end{aligned} \tag{3.15}$$

and

$$\text{(a)} \quad L_a{}^c{}_b = 0,$$
$$\text{(b)} \quad F_a{}^c{}_b = \frac{1}{2}\, g^{cd} \left(\frac{\delta g_{da}}{\delta x^b} + \frac{\delta g_{db}}{\delta x^a} - \frac{\delta g_{ab}}{\delta x^d} \right), \tag{3.16}$$

respectively.

Proof. First, by using (3.4) we obtain

$$\frac{\partial g_{ab}}{\partial y^c} = \frac{\partial g_{ac}}{\partial y^b} = \frac{\partial g_{bc}}{\partial y^a} \cdot \tag{3.17}$$

Then (3.15) follows from (3.1.25) by using (3.17). Finally, (3.16) is a consequence of (3.1.26). ∎

According to Matsumoto [Mat86], p.120, the following four Finsler connections play an important role in studying Finsler geometry:

– The **Cartan connection** $C\Gamma = (F_a{}^c{}_b, G^c_a, C_a{}^c{}_b)$,

– The **Rund connection** $R\Gamma = (F_a{}^c{}_b, G^c_a, 0)$,

– The **Berwald connection** $B\Gamma = (G_a{}^c{}_b, G^c_a, 0)$,

– The **Hashiguchi connection** $H\Gamma = (G_a{}^c{}_b, G^c_a, C_a{}^c{}_b)$.

By a **Finsler connection** we understand a pair (HTM°, ∇) where HTM° is the canonical transversal distribution locally spanned by $\left\{ \frac{\delta}{\delta x^a} \right\}$ from (3.6) (and hence given by G^b_a) and ∇ is a linear connection on VTM° or HTM°. Comparing the local coefficients of the above four Finsler connections with the local coefficients of the Vrănceanu connection presented in Proposition 3.1 we obtain the following interesting result.

Theorem 3.2.

(i) *The linear connections which determine the Hashiguchi and Rund connections coincide with the intrinsic connections D and $D^\perp$, that is, they are the restrictions of the Vrănceanu connection to VTM° and HTM° respectively.*

(ii) *The pair $(C\Gamma, B\Gamma)$ of Cartan connection and Berwald connection determines the Vrănceanu connection and viceversa.*

Therefore, the Vrănceanu connection induced by the Levi–Civita connection on (TM°, G) is equivalent to each of the two pairs of Finsler connections $(H\Gamma, R\Gamma)$ and $(C\Gamma, B\Gamma)$ on the Finsler manifold $\mathbb{F}^m = (M, F)$. Also, we remark that the Rund and Hashiguchi connections are naturally induced by the Vrănceanu connection on HTM° and VTM° respectively, which is not the case for the Cartan and Berwald connections. However, for the Cartan connection we have the following important property.

Theorem 3.3. (Bejancu–Farran [BF00b]). *The Cartan connection on the Finsler manifold* $\mathbb{F}^m = (M, F)$ *is the projection of the Levi–Civita connection* $\widetilde{\nabla}$ *of* (TM°, G) *on* VTM°. *That is to say that we have*

$$\begin{aligned} &\text{(a)} \quad G\left(\widetilde{\nabla}_{\frac{\partial}{\partial y^b}} \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^c}\right) = C_a{}^d{}_b g_{dc}, \\ &\text{(b)} \quad G\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^b}} \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^c}\right) = F_a{}^d{}_b g_{dc}. \end{aligned} \tag{3.18}$$

Moreover, we can prove the following.

Theorem 3.4. *The Levi–Civita connection* $\widetilde{\nabla}$ *on* (TM°, G) *is locally expressed as follows*

$$\begin{aligned} &\text{(a)} \ \widetilde{\nabla}_{\frac{\delta}{\delta x^b}} \frac{\delta}{\delta x^a} = -\left(C_a{}^c{}_b + \frac{1}{2} R^c{}_{ab}\right) \frac{\partial}{\partial y^c} + F_a{}^c{}_b \frac{\delta}{\delta x^c}, \\ &\text{(b)} \ \widetilde{\nabla}_{\frac{\partial}{\partial y^b}} \frac{\partial}{\partial y^a} = C_a{}^c{}_b \frac{\partial}{\partial y^c} - \frac{1}{2} g_{ab|d}\, g^{dc} \frac{\delta}{\delta x^c}, \\ &\text{(c)} \ \widetilde{\nabla}_{\frac{\delta}{\delta x^b}} \frac{\partial}{\partial y^a} = F_a{}^c{}_b \frac{\partial}{\partial y^c} + \left(C_a{}^c{}_b + \frac{1}{2} g_{ad}\, R^d{}_{eb}\, g^{ec}\right) \frac{\delta}{\delta x^c} \\ &\qquad = \widetilde{\nabla}_{\frac{\partial}{\partial y^a}} \frac{\delta}{\delta x^b} + G_a{}^c{}_b \frac{\partial}{\partial y^c}, \end{aligned} \tag{3.19}$$

where the covariant derivative in (3.19b) *is the transversal Vrănceanu covariant derivative, that is, we have* (*see* (3.1.41b))

$$g_{ab|d} = \frac{\delta g_{ab}}{\delta x^d} - g_{cb}\, G_a{}^c{}_d - g_{ac}\, G_b{}^c{}_d. \tag{3.20}$$

Proof. By using (1.5.10) for the pair $(\widetilde{\nabla}, G)$ and (3.11) we deduce that

$$G\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^b}} \frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^c}\right) = -\frac{1}{2}\left(\frac{\partial g_{ab}}{\partial y^c} + g_{cd}\, R^d{}_{ab}\right),$$

and

$$G\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^b}} \frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^c}\right) = \frac{1}{2}\left(\frac{\delta g_{ac}}{\delta x^b} + \frac{\delta g_{bc}}{\delta x^a} - \frac{\delta g_{ab}}{\delta x^c}\right).$$

Then taking into account (3.15a) and (3.16b) we obtain (3.19a). By similar calculations, using (1.5.10), (3.11) and (3.18), one can deduce (3.19b) and (3.19c). ■

Since G is parallel with respect to $\widetilde{\nabla}$, we have

$$G\left(\widetilde{\nabla}_{\frac{\partial}{\partial y^b}}\frac{\partial}{\partial y^a},\frac{\delta}{\delta x^c}\right)+G\left(\frac{\partial}{\partial y^a},\widetilde{\nabla}_{\frac{\partial}{\partial y^b}}\frac{\delta}{\delta x^c}\right)=0.$$

Then we use (3.19b) and the second equality in (3.19c) to obtain

$$g_{ab|c}=2\left(F_b{}^d{}_c-G_b{}^d{}_c\right)g_{ad}. \tag{3.21}$$

Thus (3.19b) becomes

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^b}}\frac{\partial}{\partial y^a}=C_a{}^c{}_b\,\frac{\partial}{\partial y^c}+g_{ae}\left(G_b{}^e{}_d-F_b{}^e{}_d\right)g^{dc}\,\frac{\delta}{\delta x^c}\cdot \tag{3.22}$$

Next, let us consider the Schouten–Van Kampen connection ∇° induced by $\widetilde{\nabla}$ on (TM°, G). First, since both distributions VTM° and HTM° are parallel with respect to ∇° we put

$$\begin{aligned}
&\text{(a)}\quad \nabla^\circ_{\frac{\partial}{\partial y^b}}\frac{\partial}{\partial y^a}=C^\circ{}_a{}^c{}_b\,\frac{\partial}{\partial y^c}\,,\\
&\text{(b)}\quad \nabla^\circ_{\frac{\delta}{\delta x^b}}\frac{\partial}{\partial y^a}=G^\circ{}_a{}^c{}_b\,\frac{\partial}{\partial y^c}\,,\\
&\text{(c)}\quad \nabla^\circ_{\frac{\partial}{\partial y^b}}\frac{\delta}{\delta x^a}=L^\circ{}_a{}^c{}_b\,\frac{\delta}{\delta x^c}\,,\\
&\text{(d)}\quad \nabla^\circ_{\frac{\delta}{\delta x^b}}\frac{\delta}{\delta x^a}=F^\circ{}_a{}^c{}_b\,\frac{\delta}{\delta x^c}\cdot
\end{aligned} \tag{3.23}$$

Then by using (3.2.13) and Theorem 3.4 we obtain the following.

Proposition 3.5. *The local coefficients of the induced connections* ∇ *and* $\nabla^\perp$ *on* VTM° *and* HTM° *with respect to the semi–holonomic frame field* $\left\{\frac{\delta}{\delta x^a},\frac{\partial}{\partial y^a}\right\}$ *are given by*

$$\begin{aligned}
&\text{(a)}\quad C^\circ{}_a{}^c{}_b=C_a{}^c{}_b,\\
&\text{(b)}\quad G^\circ{}_a{}^c{}_b=F_a{}^c{}_b,
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&\text{(a)}\quad L^\circ{}_a{}^c{}_b=C_a{}^c{}_b+\frac{1}{2}\,g_{bd}\,R^d{}_{ea}\,g^{ec},\\
&\text{(b)}\quad F^\circ{}_a{}^c{}_b=F_a{}^c{}_b,
\end{aligned} \tag{3.25}$$

respectively, where $C_a{}^c{}_b, F_a{}^c{}_b$ *and* $R_a{}^c{}_b$ *are given by* (3.15a), (3.16b) *and* (3.12) *respectively.*

Corollary 3.6. *The linear connection which determines the Cartan connection on* $\mathbb{F}^m=(M,F)$ *is just the induced connection* ∇ *on* VTM°*, i.e., it coincides with the restriction of the Schouten–Van Kampen connection to* VTM°*.*

Now, we recall that in Chapter 3 we presented several classes of foliations on semi–Riemannian manifolds. We have seen also that for every Finsler manifold $\mathbb{F}^m = (M, F)$ there is a natural foliation $\mathcal{F}_V$ on TM°. We will see later in this section that TM° admits some more natural foliations. So it is interesting to investigate the geometry of $\mathbb{F}^m$ when those foliations belong to any of these classes. That is to say, we will study the relationship between the geometry of the foliations on TM° on the one hand, and the geometry of $\mathbb{F}^m$ on the other hand.

First, we recall that $\mathbb{F}^m = (M, F)$ is a **Landsberg manifold** (see Bejancu–Farran, [BF00a], p.64) if the Berwald connection coincides with the Rund connection, that is, we have

$$F_a{}^c{}_b = G_a{}^c{}_b, \quad \forall\, a, b, c \in \{1, ..., m\}. \tag{3.26}$$

The next theorem gives an interesting characterization of a Landsberg manifold by means of the vertical foliation.

Theorem 3.7. *A Finsler manifold $\mathbb{F}^m = (M, F)$ is a Landsberg manifold if and only if the vertical foliation $\mathcal{F}_V$ on the Riemannian manifold (TM°, G) is totally geodesic.*

Proof. Taking into account (3.26), (3.24) and (3.15) we deduce that $\mathbb{F}^m$ is a Landsberg manifold if and only if the induced connection ∇ coincides with the intrinsic connection D on VTM°. Then the assertion of the theorem follows from (i) and (ii) of Theorem 3.4.2. ■

Now, we recall that a Finsler manifold $\mathbb{F}^m = (M, F)$ is a Riemannian manifold, if and only if the Finsler metric (3.4) depends on (x^a) alone, that is,

$$\frac{\partial g_{ab}}{\partial y^c} = 0, \quad \forall\, a, b, c \in \{1, ..., m\}. \tag{3.27}$$

Taking into account (3.15a), we deduce that $\mathbb{F}^m$ is Riemannian if and only if

$$C_a{}^c{}_b = 0, \quad \forall\, a, b, c \in \{1, ..., m\}. \tag{3.28}$$

The vertical foliation on (TM°, G) can be used to characterize Riemannian manifolds as follows.

Theorem 3.8. *A Finsler manifold $\mathbb{F}^m = (M, F)$ is a Riemannian manifold if and only if the Sasaki–Finsler metric G on TM° is bundle–like for the vertical foliation.*

Proof. It follows by using (3.27) and Theorem 3.3.2. ■

On the other hand, any Riemannian manifold is a Landsberg manifold. Then, from Theorems 3.7 and 3.8, we deduce the following.

Corollary 3.9. *Let (M, g) be a Riemannian manifold. Then the vertical foliation $\mathcal{F}_V$ is totally geodesic on (TM, G) and G is bundle–like for $\mathcal{F}_V$.*

Next, we consider two globally defined vector fields on TM° which have an important impact on Finsler (Riemannian) geometry. We define them as follows:

$$\begin{aligned} &(a) \quad L = y^a \frac{\partial}{\partial y^a}, \\ &(b) \quad L^* = y^a \frac{\delta}{\delta x^a}, \end{aligned} \tag{3.29}$$

where $\left\{ \frac{\delta}{\delta x^a} \right\}$, $a \in \{1, ..., m\}$, are given by (3.6). Both L and L^* are globally defined on TM° since, with respect to the coordinate transformation (3.1), we have (3.3) and

$$\frac{\delta}{\delta x^a} = J_a^b(x) \frac{\delta}{\delta \widetilde{x}^b} \cdot \tag{3.30}$$

L is known as the **Liouville vector field** on TM°. As L^* lies in the transversal distribution to the vertical foliation on TM° we call it the **transversal Liouville vector field**. Now, we denote by $\mathcal{L}$ and $\mathcal{L}^*$ the line fields spanned by L and L^* and call them the **Liouville distribution** and the **transversal Liouville distribution** on TM°. Now, we need some identities from Finsler geometry. First, because most of the geometric objects from Finsler geometry are positive homogeneous of certain degree we present the following.

Theorem 3.10. (Euler's Theorem). *A smooth function $f(y^1, ..., y^m)$ on $\mathbb{R}^m \backslash \{0\}$ is positively homogeneous of degree r if and only if it satisfies the condition*

$$y^a \frac{\partial f}{\partial y^a} = rf. \tag{3.31}$$

Next, from the definition of the Finsler manifold we see that $g_{ab}(x, y)$, $G^a(x, y)$ and $G_b^a(x, y)$ are positively homogeneous of degrees 0, 2 and 1 respectively. Thus, by using (3.31), (3.7) and (3.15) we obtain

$$\begin{aligned} &(a) \quad y^a C_a{}^c{}_b = 0, \\ &(b) \quad y^a G_a^b = 2G^b, \\ &(c) \quad y^a G_a{}^b{}_c = G_c^b. \end{aligned} \tag{3.32}$$

Now, we define the functions γ_{abc} and $\gamma_b{}^a{}_c$ by

$$\begin{aligned} &(a) \quad \gamma_{abc} = \frac{1}{2} \left(\frac{\partial g_{ba}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ac}}{\partial x^b} \right), \\ &(b) \quad \gamma_b{}^a{}_c = g^{ad} \gamma_{bdc}, \end{aligned} \tag{3.33}$$

and by direct calculations using (3.4), (3.5) and (3.33) we deduce that

$$\begin{aligned} &\text{(a)} \quad G^a = \frac{1}{2}\,\gamma_b{}^a{}_c\, y^b y^c, \\ &\text{(b)} \quad g_{ad}\, G^a = \frac{1}{2}\,\gamma_{bdc}\, y^b y^c. \end{aligned} \tag{3.34}$$

Then, differentiating (3.33) with respect to y^e and taking into account (3.15a) we infer that

$$\frac{\partial \gamma_{abc}}{\partial y^e} = \frac{\partial}{\partial x^c}\left(g_{ed} C_b{}^d{}_a\right) + \frac{\partial}{\partial x^a}\left(g_{ed} C_b{}^d{}_c\right) - \frac{\partial}{\partial x^b}\left(g_{ed} C_a{}^d{}_c\right). \tag{3.35}$$

By contracting (3.35) with $y^a y^c$ and using (3.32a) we obtain

$$\frac{\partial \gamma_{abc}}{\partial y^e}\, y^a y^c = 0. \tag{3.36}$$

We differentiate (3.34b) with respect to y^e and by using (3.15a), (3.7) and (3.36) we deduce that

$$G_b^c = y^a\, \gamma_a{}^c{}_b - 2G^a C_a{}^c{}_b. \tag{3.37}$$

Now, by direct calculations using (3.6), (3.33) and (3.15a), (3.16b) becomes

$$F_a{}^c{}_b = \gamma_a{}^c{}_b - G_a^d C_d{}^c{}_b - G_b^d C_d{}^c{}_a + G_e^d g^{ce} g_{df} C_a{}^f{}_b. \tag{3.38}$$

Finally, by contracting (3.38) by y^a and using (3.32b), (3.32a) and (3.37) we obtain

$$y^a F_a{}^c{}_b = G_b^c. \tag{3.39}$$

We also note that $C_a{}^c{}_b, G_a{}^c{}_b$ and $F_a{}^c{}_b$ are symmetric with respect to (ab). Next, by using (3.6) we infer that

$$\frac{\delta y^a}{\delta x^b} = -G_b^a. \tag{3.40}$$

To compute the covariant derivative of L with respect to the induced connection ∇ on VTM°, we note that ∇ is the restriction of the Schouten–Van Kampen connection ∇° to VTM°. Then, by using (3.40), (3.23b), (3.24b) and (3.39) we deduce that

$$\begin{aligned} \nabla_{\frac{\delta}{\delta x^a}} L = \nabla^\circ_{\frac{\delta}{\delta x^a}}\, y^b \frac{\partial}{\partial y^b} &= \frac{\delta y^b}{\delta x^a}\, \frac{\partial}{\partial y^b} + y^c\, \nabla^\circ_{\frac{\delta}{\delta x^a}} \frac{\partial}{\partial y^c} \\ &= \left(-G_a^b + y^c F_c{}^b{}_a\right) \frac{\partial}{\partial y^b} = 0. \end{aligned} \tag{3.41}$$

Similarly, by using (3.23a), (3.24a) and (3.32a) we obtain

$$\nabla_{\frac{\partial}{\partial y^a}} L = \nabla^{\circ}_{\frac{\partial}{\partial y^a}} y^b \frac{\partial}{\partial y^b} = \frac{\partial}{\partial y^a} \cdot \tag{3.42}$$

Now, denote by R the curvature tensor field of ∇. Then, by direct calculations using (1.2.17), (3.11b), (3.41) and (3.42), we infer that

$$R\left(\frac{\delta}{\delta x^c}, \frac{\delta}{\delta x^b}\right) L = R^a{}_{bc} \frac{\partial}{\partial y^a} \cdot \tag{3.43}$$

Thus, if we put

$$R\left(\frac{\delta}{\delta x^c}, \frac{\delta}{\delta x^b}\right) \frac{\partial}{\partial y^a} = R_a{}^d{}_{bc} \frac{\partial}{\partial y^d}, \tag{3.44}$$

then (3.43) implies

$$y^a R_a{}^d{}_{bc} = R^d{}_{bc}. \tag{3.45}$$

In the terminology of Finsler geometry $R_a{}^d{}_{bc}$ are the local components of the h–curvature tensor field of the Cartan connection (see Matsumoto, [Mat86], p.114).

We denote by H and V the projection morphisms of TTM° on HTM° and VTM° respectively. Then, by (1.6.3), we have

$$\begin{aligned} G(\widetilde{R}(X,Y)VZ,VU) = {} & G(R(X,Y)VZ,VU) \\ & + G(h(X,VZ),h(Y,VU)) \\ & - G(h(Y,VZ),h(X,VU)), \end{aligned} \tag{3.46}$$

where $\widetilde{R}$ is the curvature tensor field of the Levi–Civita connection $\widetilde{\nabla}$ on (TM°, G) and h is given by (see (1.5.20a))

$$h(X,VZ) = H\widetilde{\nabla}_X VZ, \quad \forall\, X, Z \in \Gamma(TTM).$$

Further, we put

$$R_{abcd} = G\left(R\left(\frac{\delta}{\delta x^d}, \frac{\delta}{\delta x^c}\right) \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) = g_{be} R_a{}^e{}_{cd}, \tag{3.47}$$

and from (3.46) we deduce that

$$R_{abcd} + R_{bacd} = 0. \tag{3.48}$$

Finally, we put

$$R_{bcd} = g_{ba}\, R^a{}_{cd}, \tag{3.49}$$

and, by using (3.45) and (3.47), we obtain

$$R_{bcd} = y^a\, R_{abcd}. \tag{3.50}$$

Thus, from (3.48) and (3.50) we deduce the following important identity

$$y^b R_{bcd} = 0. \tag{3.51}$$

Now, we decompose each $X \in \Gamma(TTM^\circ)$ as follows

$$X = VX + HX = (VX)^a \frac{\partial}{\partial y^a} + (HX)^a \frac{\delta}{\delta x^a} \cdot \tag{3.52}$$

Then we prove the following.

Lemma 3.11. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold. Then for any $X \in \Gamma(TTM^\circ)$ we have*

$$\begin{aligned}
&\text{(a) } \widetilde{\nabla}_X L = VX,\\
&\text{(b) } \widetilde{\nabla}_X L^* = \frac{1}{2}(HX)^b R^a{}_{bc}\, y^c \frac{\partial}{\partial y^a}\\
&\qquad + \left((VX)^a + \frac{1}{2}(VX)^c R_{cbd}\, y^d g^{ba}\right) \frac{\delta}{\delta x^a},\\
&\text{(c) } \nabla^\circ_X L = \nabla^*_X L = VX,\\
&\text{(d) } \nabla^\circ_X L^* = \left((VX)^a + \frac{1}{2}(VX)^c R_{cbd}\, y^d g^{ba}\right) \frac{\delta}{\delta x^a},\\
&\text{(e) } \nabla^*_X L^* = (VX)^a \frac{\delta}{\delta x^a},
\end{aligned} \tag{3.53}$$

where $\widetilde{\nabla}$ is the Levi–Civita connection on (TM°, G) and ∇^* and ∇° are the Vrănceanu and Schouten–Van Kampen connections defined by $\widetilde{\nabla}$ with respect to the decomposition (3.8).

Proof. First, by direct calculations, using (3.52), (3.29a) and (3.40), we deduce that

$$\begin{aligned}
\widetilde{\nabla}_X L &= (VX)^a \frac{\partial}{\partial y^a} - (HX)^b G^a_b \frac{\partial}{\partial y^a}\\
&\quad + (VX)^b y^a \widetilde{\nabla}_{\frac{\partial}{\partial y^b}} \frac{\partial}{\partial y^a} + (HX)^b y^a \widetilde{\nabla}_{\frac{\delta}{\delta x^b}} \frac{\partial}{\partial y^a} \cdot
\end{aligned}$$

Then we replace the above covariant derivatives by their expressions from (3.22) and (3.19c) and using (3.32a), (3.32c), (3.39) and (3.51) we obtain (3.53a). Similar calculations lead us to (3.53b). Next, from (3.41) and (3.42) we infer that

$$\nabla^\circ_X L = VX.$$

Also, by using (3.1.12), (3.29a), (3.11a), (3.15b), (3.32a), (3.32c) and (3.53a) we deduce that

$$\begin{aligned}
\nabla^*_X L &= V\widetilde{\nabla}_{VX} L + V[HX, L]\\
&= VX + (HX)^a (y^b G_b{}^c{}_a - G^c_a) \frac{\partial}{\partial y^c} = VX.
\end{aligned}$$

This completes the proof for (3.53c). By similar calculations for the Schouten–Van Kampen and Vrănceanu covariant derivatives of L^* we obtain (3.53d) and (3.53e). ∎

As a consequence of (3.53a) and (3.53b) we obtain the following.

Corollary 3.12. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold and $\widetilde{\nabla}$ be the Levi–Civita connection on (TM°, G). Then we have*

$$\begin{aligned} &\text{(a) } \widetilde{\nabla}_{VX} L = VX, \quad &&\text{(b) } \widetilde{\nabla}_{HX} L = 0, \forall X \in \Gamma(TTM), \\ &\text{(c) } \widetilde{\nabla}_{L} L = L, &&\text{(d) } \widetilde{\nabla}_{L^*} L^* = 0. \end{aligned} \tag{3.54}$$

Moreover, we prove the following theorem.

Theorem 3.13. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold. Then the vector fields L and L^* determine two totally geodesic foliations on (TM°, G).*

Proof. Denote by h_L the second fundamental form of the foliation determined by L. Then by using (3.2.5) and (3.54c) we obtain

$$G(h(L, L), X) = G(\widetilde{\nabla}_L L, X) = G(L, X) = 0,$$

for any vector field X on TM° that is orthogonal to L. Hence the foliation determined by L is totally geodesic. Similarly, by using (3.2.5) for the second fundamental form of $\mathcal{L}^*$ and (3.54d) we deduce that L^* determines a totally geodesic foliation too. ∎

From (3.54d) we see that the integral curves of L^* are geodesics in (TM°, G). The next proposition says that L^* could give us more information about the geometry of the Finsler manifold $\mathbb{F}^m$ itself. Actually, Theorem 3.23 is a result in this direction.

Proposition 3.14. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold. Then the projection of an integral curve of the transversal Liouville vector field L^* on M is a geodesic of $\mathbb{F}^m$.*

Proof. By using (3.29b) and (3.6), we deduce that an integral curve $\Gamma : x^a = x^a(t)$, $y^a = y^a(t)$, $t \in I$ of L^* is a solution of the differential system

$$\frac{dx^a}{dt} = y^a, \quad \frac{dy^a}{dt} = -y^b G^a_b(x, y).$$

Thus the projection $C : x^a = x^a(t)$, $t \in I$, of Γ on M is a solution of the system

$$\frac{d^2x^a}{dt^2} + G^a_b(x(t), x'(t))\,\frac{dx^b}{dt} = 0.$$

Hence, according to Matsumoto [Mat86], p. 281, C is a geodesic of $\mathbb{F}^m$. ∎

Apart from the foliations with tangent distributions VTM°, $\mathcal{L}$ and $\mathcal{L}^*$, on the open submanifold TM° of the tangent bundle TM of a Finsler manifold $\mathbb{F}^m = (M, F)$ there are three more foliations. To introduce them we denote by $\mathcal{L}'$ and $\mathcal{L}^\perp$ the complementary orthogonal distributions to $\mathcal{L}$ in VTM° and TTM° respectively. Then we prove the following.

Theorem 3.15. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold. Then the distributions $\mathcal{L}^\perp, \mathcal{L}'$ and $\mathcal{L} \oplus \mathcal{L}^*$ are integrable.*

Proof. First, let $X, Y \in \Gamma(\mathcal{L}^\perp)$. Then, by using the properties of the Levi–Civita connection $\widetilde{\nabla}$ on (TM°, G) and (3.53a), we obtain

$$\begin{aligned} G([X,Y], L) &= G(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X, L) = G(X, \widetilde{\nabla}_Y L) - G(Y, \widetilde{\nabla}_X L) \\ &= G(VX, VY) - G(VY, VX) = 0. \end{aligned} \tag{3.55}$$

Hence $\mathcal{L}^\perp$ is integrable. Now, we take $X, Y \in \Gamma(\mathcal{L}')$. Then since VTM° is integrable and $\mathcal{L}'$ is a vector subbundle, we conclude that $[X, Y] \in \Gamma(VTM^\circ)$. Then, from (3.55), we deduce that $[X, Y] \in \Gamma(\mathcal{L}')$, that is, $\mathcal{L}'$ is integrable too. Finally, by direct calculations, using (3.53a), (3.53b) and (3.51), we infer that

$$\text{(a) } \widetilde{\nabla}_{L^*} L = 0 \quad \text{and} \quad \text{(b) } \widetilde{\nabla}_L L^* = L^*. \tag{3.56}$$

Thus

$$[L, L^*] = \widetilde{\nabla}_L L^* - \widetilde{\nabla}_{L^*} L = L^* \in \Gamma(\mathcal{L} \oplus \mathcal{L}^*),$$

and hence $\mathcal{L} \oplus \mathcal{L}^*$ is an integrable 2–distribution. ∎

Moreover, from (3.54c), (3.54d) and (3.56), we deduce the following.

Corollary 3.16. *The foliation determined by the distribution $\mathcal{L} \oplus \mathcal{L}^*$ is totally geodesic on (TM°, G).*

Next, we want to show that the leaves of the foliations determined by $\mathcal{L}^\perp$ and $\mathcal{L}'$ are defined by means of the fundamental function F of $\mathbb{F}^m$ and to point out some interesting properties of them. First, we recall that F is a positive–valued smooth function on TM°. Moreover, it has no critical points. Indeed, by contracting (3.4) with $y^b y^c$ and using Euler theorem, we deduce that

$$F^2(x, y) = g_{ab}(x, y) y^a y^b. \tag{3.57}$$

Differentiating (3.57) with respect to y^c and using (3.17) and (3.32a) we obtain

$$\frac{\partial F}{\partial y^c} = g_{cb}(x,y)\frac{y^b}{F} \neq 0, \quad \text{on } TM^\circ. \tag{3.58}$$

Hence F defines a foliation $\mathcal{F}_F$ of TM° whose leaves are connected components of level hypersurfaces of F (see Example 2.1.1). We denote by $IM(c)$ a leaf of $\mathcal{F}_F$ given by the equation

$$F(x,y) = c, \tag{3.59}$$

where c is a positive constant. Now, we recall that the **gradient** of F is a vector field denoted by $\operatorname{grad} F$ and defined by (cf. O'Neill [O83], p.85)

$$G(\operatorname{grad} F, X) = X(F), \quad \forall\, X \in \Gamma(TTM^\circ). \tag{3.60}$$

Moreover, $\operatorname{grad} F$ is the normal vector field to the leaf $IM(c)$ given by (3.59). Then, by using (3.60) and the decomposition (3.52), we deduce that X is tangent to $IM(c)$ if and only if

$$(VX)^a\,\frac{\partial F}{\partial y^a} + (HX)^a\,\frac{\delta F}{\delta x^a} = 0. \tag{3.61}$$

Now, we express (3.57) as follows

$$F^2 = G(L,L). \tag{3.62}$$

Apply $\dfrac{\delta}{\delta x^a}$ to (3.62) and by using (3.54b) and (1.5.9) for $(\widetilde{\nabla}, G)$ we obtain

$$F\,\frac{\delta F}{\delta x^a} = 2G\left(\widetilde{\nabla}_{\frac{\delta}{\delta x^a}} L, L\right) = 0,$$

which implies

$$\frac{\delta F}{\delta x^a} = 0, \quad \forall\, a \in \{1, ..., m\}. \tag{3.63}$$

Thus, taking into account (3.61) and (3.63), we deduce that a vector field X is tangent to $IM(c)$ if and only if

$$(VX)^a\,\frac{\partial F}{\partial y^a} = 0, \tag{3.64}$$

which, via (3.58), is equivalent to

$$g_{ab}(VX)^a y^b = 0. \tag{3.65}$$

As (3.65) also represents the condition for X to be orthogonal to L, we can state the following.

Proposition 3.17.

(i) *The foliation $\mathcal{F}_F$ determined by the level hypersurfaces of the fundamental function F of the Finsler manifold $\mathbb{F}^m$ is just the foliation determined by the distribution $\mathcal{L}^\perp$.*

(ii) *The Liouville vector field L is orthogonal to the foliation $\mathcal{F}_F$.*
(iii) *The transversal Liouville vector field L^* is tangent to the foliation $\mathcal{F}_F$.*

As $\mathcal{F}_F$ is determined by the fundamental function F of $\mathbb{F}^m$ we call it the **fundamental foliation** on (TM°, G).

Next, we consider a fixed point $x_0 = (x_0^a)$ on M and the hypersurface $I_{x_0}M(c)$, $c > 0$, in $T_{x_0}M$ given by the equation

$$F(x_0, y) = c, \quad \forall\, y \in T_{x_0}M. \tag{3.66}$$

According to Matsumoto [Mat86], p.105, $I_{x_0}M(1)$ is called the **indicatrix** of the Finsler manifold $\mathbb{F}^m$ at x_0. In general, we say that $I_{x_0}M(c)$ is the c–**indicatrix** of $\mathbb{F}^m$ at x_0. To state some properties of $I_{x_0}M(c)$ we consider $T_{x_0}M$ as a Riemannian manifold with the Riemannian metric $g_{x_0} = (g_{ab}(x_0, y))$. A vector field X tangent to $T_{x_0}M$ is expressed as follows

$$X = X^a(x_0, y)\, \frac{\partial}{\partial y^a}\bigg|_{(x_0,y)}, \quad y \in T_{x_0}M.$$

Then, by a similar reason as for the leaf $IM(c)$ of $\mathcal{F}_F$, we deduce that X is tangent to $I_{x_0}M(c)$ if and only if

$$\begin{aligned} &\text{(a)} \quad X^a(x_0, y)\, \frac{\partial F}{\partial y^a}\,(x_0, y) = 0, \quad \text{or} \\ &\text{(b)} \quad g_{ab}(x_0, y) X^a(x_0, y) y^b = 0. \end{aligned} \tag{3.67}$$

From (3.67b) it follows that the Liouville vector field L is the normal vector field to each hypersurface $I_{x_0}M(c)$ in the Riemannian manifold $(T_{x_0}M, g_{x_0})$. Moreover $\{I_{x_0}M(c)\}_{c\in\mathbb{R}_+}$ are level hypersurfaces for the function $F_{x_0}(y) = F(x_0, y)$ on $T_{x_0}M$ which does not have critical points (see (3.58)). Hence the set of all c–indicatrices at x_0 determines a foliation of codimension 1 of the m–dimensional Riemannian manifold $(T_{x_0}M, g_{x_0})$. We denote it by $I_{x_0}M$ and call it the **indicatrix foliation** at x_0.

Next, from (3.62) we obtain $G(\ell, \ell) = 1$, where ℓ is the **unit Liouville vector field**, that is, we have

$$\ell = \frac{1}{F}\, L = \ell^a\, \frac{\partial}{\partial y^a}, \quad \text{where } \ell^a = \frac{y^a}{F} \cdot \tag{3.68}$$

Let ∇' and ∇'' be the Levi–Civita connections on $(T_{x_0}M, g_{x_0})$ and $(I_{x_0}M(c), g_{x_0})$ respectively. Then we have

$$\begin{aligned} &\text{(a)} \quad \widetilde{\nabla}_X Y = \nabla'_X Y + h_{x_0}(X, Y), \\ &\text{(b)} \quad \nabla'_X Y = \nabla''_X Y + B(X, Y)\ell, \end{aligned} \tag{3.69}$$

where $\widetilde{\nabla}$ is the Levi–Civita connection on (TM°, G), and h_{x_0} and B are the second fundamental forms of $T_{x_0}M$ and $I_{x_0}M(c)$ as submanifolds of TM°

and $T_{x_0}M$ respectively. Then, by using (3.69), (3.53a) and properties of $\widetilde{\widetilde{\nabla}}$, we obtain

$$\begin{aligned} B(X,Y) &= g_{x_0}(\nabla'_X Y, \ell) \\ &= G(\widetilde{\widetilde{\nabla}}_X Y, \ell) = -\,G(Y, \widetilde{\widetilde{\nabla}}_X \ell) \\ &= -\,G\left(Y, X\left(\frac{1}{F}\right) L + \frac{1}{F}\,\widetilde{\widetilde{\nabla}}_X L\right) \\ &= -\frac{1}{F}\,G(X,Y), \end{aligned}$$

for any $X, Y \in \Gamma(TI_{x_0}M(c))$. This means that any c–indicatrix at x_0 is totally umbilical immersed in $T_{x_0}(M)$. Hence the indicatrix foliation at x_0 is a totally umbilical foliation of $T_{x_0}M$. Finally, since L is the normal vector field to each c–indicatrix we deduce that the leaves of $\mathcal{L}'$ (see Theorem 3.15) are c–indicatrices. Summing up the above discussion we state the following.

Proposition 3.18. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold. Then we have the assertions:*

(i) *At any point $x \in M$ the indicatrix foliation I_xM is a totally umbilical foliation of (T_xM, g_x).*
(ii) *The leaves of the foliation $\mathcal{F}_{\mathcal{L}'}$ determined by the integrable distribution $\mathcal{L}'$ are c–indicatrices of $\mathbb{F}^m$.*
(iii) *Each leaf of $\mathcal{F}_{\mathcal{L}'}$ is a totally umbilical submanifold of a leaf of the vertical foliation $\mathcal{F}_V$.*

Now, we consider a leaf $IM(c)$ of the fundamental foliation $\mathcal{F}_F$ on (TM°, G). Then, by (3.59) and (3.66) we deduce that

$$IM(c) = \bigcup_{x \in M} I_xM(c).$$

Thus we may call $IM(c)$ the c–**indicatrix bundle** over M. We show in what follows an interesting relationship between the geometry of the c–indicatrix bundle over M and that of the curvature of M. To this end we start with the identity (cf. Bejancu–Farran [BF00a], p.52)

$$R_{abc} + R_{bca} + R_{cab} = 0. \tag{3.70}$$

Contracting (3.70) by y^c and using (3.51) we obtain

$$R_{bca} y^c = R_{acb} y^c,$$

since R_{abc} is skew–symmetric with respect to the pair (bc). Hence

$$R_{ab} = R_{acb} y^c, \quad a, c \in \{1, ..., m\}, \tag{3.71}$$

are the components of a symmetric Finsler tensor field of type $(0,2)$ on TM° (cf. Bejancu–Farran [BF00a], p.13). We also consider the **angular metric** h_{ab} of $\mathbb{F}^m$ (cf. Matsumoto [Mat86], p.101)

$$h_{ab} = g_{ab} - \ell_a \ell_b, \tag{3.72}$$

where we set

$$\ell_a = g_{ab} \ell^b = g_{ab} \frac{y^b}{F} \cdot \tag{3.73}$$

Finally, we define the symmetric Finsler tensor field $\Lambda = (\Lambda_{ab})$ given by

$$\Lambda_{ab} = R_{ab} - h_{ab}. \tag{3.74}$$

We consider Λ as a symmetric bilinear $F(TM^\circ)$–form on $\Gamma(HTM^\circ)$ and call it the **curvature–angular form**.

Proposition 3.19. *For any $X \in \Gamma(HTM^\circ)$ we have*

$$\Lambda(L^*, X) = 0, \tag{3.75}$$

that is, the curvature–angular form is degenerate.

Proof. By using (3.74), (3.71), (3.51), (3.72), (3.73) and (3.57) we obtain

$$\begin{aligned} \Lambda_{ab} y^a &= y^a R_{acb} y^c - y^a g_{ab} + y^a \ell_a \ell_b \\ &= -F\ell_b + y^a g_{ac} \frac{y^c}{F} \ell_b \\ &= -F\ell_b + F\ell_b = 0, \end{aligned}$$

which proves (3.75). ∎

Next, we consider the foliation determined by the transversal Liouville vector field L^*. By Theorem 3.13 we have seen that this foliation is totally geodesic on (TM°, G). Moreover, by (3.54d) we deduce that it is totally geodesic on any c–indicatrix bundle $(IM(c), G)$. Here and in the sequel, we denote by the same symbol G the induced Riemannian metric on $IM(c)$ by the Sasaki–Finsler metric G on TM°, and call it the **Sasaki–Finsler metric** on $IM(c)$. The next theorem gives an interesting condition for G to be bundle–like for the above foliation.

Theorem 3.20. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold and $IM(c)$ be a c–indicatrix bundle over M. Then the following assertions are equivalent:*

(i) *The Sasaki–Finsler metric G on $IM(c)$ is bundle–like for the foliation $\mathcal{F}$ determined by the transversal Liouville vector field L^* on $IM(c)$.*
(ii) *The curvature–angular form Λ vanishes identically on $IM(c)$.*

Proof. Let $\overline{\nabla}$ be the Levi–Civita connection on $(IM(c), G)$ and $\mathcal{L}''$ be the complementary orthogonal distribution to $\mathcal{L}^*$ in HTM°. Here all the vector bundles are considered over $IM(c)$. Then $\mathcal{L}^\perp = \mathcal{L}' \oplus \mathcal{L}'' \oplus \mathcal{L}^*$ is the tangent bundle of $IM(c)$. From assertion (iii) of Theorem 3.3.3 we deduce that G is bundle–like for $\mathcal{F}$ if and only if

$$G(\overline{\nabla}_X L^*, Y) + G(\overline{\nabla}_Y L^*, X) = 0, \quad \forall X, Y \in \Gamma(\mathcal{L}' \oplus \mathcal{L}''),$$

which is equivalent to

$$G(\widetilde{\nabla}_X L^*, Y) + G(\widetilde{\nabla}_Y L^*, X) = 0, \quad \forall X, Y \in \Gamma(\mathcal{L}' \oplus \mathcal{L}''). \tag{3.76}$$

We consider the following three cases to analyze (3.76).

Case 1. $X \in \Gamma(\mathcal{L}')$, $Y \in \Gamma(\mathcal{L}')$. Then, from (3.53b) we deduce that $\widetilde{\nabla}_X L^* \in \Gamma(HTM^\circ)$ and $\widetilde{\nabla}_Y L^* \in \Gamma(HTM^\circ)$. Thus, in this case (3.76) is identically satisfied because $\mathcal{L}'$ and HTM° are orthogonal vector bundles with respect to G.

Case 2. $X \in \Gamma(\mathcal{L}'')$, $Y \in \Gamma(\mathcal{L}'')$. Now, (3.53b) implies $\widetilde{\nabla}_X L^* \in \Gamma(VTM^\circ)$ and $\widetilde{\nabla}_Y L^* \in \Gamma(VTM^\circ)$, and therefore (3.76) is identically verified.

Case 3. $X \in \Gamma(\mathcal{L}')$, $Y \in \Gamma(\mathcal{L}'')$. In this case we have $X = X^a \dfrac{\partial}{\partial y^a}$ and $Y = Y^a \dfrac{\delta}{\delta x^a}$, where the local components satisfy

$$\begin{aligned} &\text{(a)} \quad g_{ab} X^a y^b = 0 \quad \text{and} \\ &\text{(b)} \quad g_{ab} Y^a y^b = 0. \end{aligned} \tag{3.77}$$

Then, by using (3.53b), (3.49) and (3.71), the condition (3.76) becomes

$$(g_{ab} - R_{ab}) X^a Y^b = 0. \tag{3.78}$$

Next, taking into account (3.72), (3.73) and (3.77), we deduce that

$$h_{ab} X^a Y^b = g_{ab} X^a Y^b. \tag{3.79}$$

Hence, by using (3.79) into (3.78) and taking into account (3.74), we obtain

$$\Lambda_{ab} X^a Y^b = 0. \tag{3.80}$$

Now, we consider the isomorphism of vector bundles

$$\Phi : \mathcal{L}' \longrightarrow \mathcal{L}''; \quad \Phi\left(X^a \frac{\partial}{\partial y^a}\right) = X^* = X^a \frac{\delta}{\delta x^a} \cdot$$

Then (3.80) is equivalent to

$$\Lambda(X^*, Y) = 0, \quad \forall X^*, Y \in \Gamma(\mathcal{L}''). \tag{3.81}$$

Finally, from (3.81) and (3.75) we conclude that (3.76) is equivalent to $\Lambda = 0$ on $IM(c)$, which completes the proof of the theorem. ∎

Apparently, the condition (3.76) seems to be weaker than the condition for L^* to be Killing vector field on $IM(c)$. However, we prove the following.

Theorem 3.21. *The transversal Liouville vector field L^* is a Killing vector field on $IM(c)$ if and only if the curvature–angular form Λ vanishes identically on $IM(c)$.*

Proof. Suppose L^* is Killing on $IM(c)$. Then we have

$$G(\widetilde{\nabla}_X L^*, Y) + G(\widetilde{\nabla}_Y L^*, X) = 0, \quad \forall\, X, Y \in \Gamma(\mathcal{L}^\perp). \tag{3.82}$$

Thus (3.76) is satisfied, and by Theorem 3.20 it follows that $\Lambda = 0$ on $IM(c)$. Conversely, if $\Lambda = 0$ on $IM(c)$, by the same theorem we deduce that (3.76) holds. To prove (3.82) we need to show that

$$G(\widetilde{\nabla}_{L^*} L^*, Y) + G(\widetilde{\nabla}_Y L^*, L^*) = 0, \quad \forall\, Y \in \Gamma(\mathcal{L}' \oplus \mathcal{L}''). \tag{3.83}$$

By (3.54d) the first term in (3.83) vanishes. Now, take $Y \in \Gamma(\mathcal{L}'')$ and from (3.53b) we see that $\widetilde{\nabla}_Y L^* \in \Gamma(VTM^\circ)$. Hence the second term in (3.83) vanishes too. Finally, take $Y \in \Gamma(\mathcal{L}')$, that is,

$$Y = Y^a \frac{\partial}{\partial y^a} \quad \text{and} \quad Y^a g_{ab} y^b = 0.$$

Then, by using again (3.53b) and taking into account that R_{abc} is skew–symmetric with respect to (bc), we obtain

$$\begin{aligned} G(\widetilde{\nabla}_Y L^*, L^*) &= \left(Y^a + \frac{1}{2} Y^c R_{cbd} y^d g^{ba} \right) y^e g_{ae} \\ &= Y^a g_{ae} y^e + \frac{1}{2} Y^c R_{cbd} y^b y^d = 0. \end{aligned}$$

Thus (3.83) is identically satisfied, and therefore L^* is a Killing vector field on $IM(c)$. ∎

Next, denote by the same symbol L^* the transversal Liouville vector at a point $(x, y) \in TM^\circ$ and consider a vector $X = X^a \dfrac{\delta}{\delta x^a} \in HTM^\circ(x, y)$ such that $\{L^*, X\}$ are linearly independent. Then the plane $\Pi = \operatorname{span}\{L^*, X\}$ is called the **horizontal flag** at (x, y) with L^* as flagpole and X as transverse edge. The **horizontal flag curvature** of the Finsler manifold $\mathbb{F}^m = (M, F)$ with respect to the horizontal flag Π is defined by (cf. Bejancu–Farran [BF00a], p.57)

$$K(x, y; \Pi) = \frac{R_{abcd} y^a X^b y^c X^d}{F^2 h_{ab} X^a X^b}, \tag{3.84}$$

where R_{abcd} and h_{ab} are the local components of the h–curvature tensor field of the Cartan connection and of the angular metric respectively. When

$K(x, y; \Pi)$ is independent of the horizontal flag Π, $\mathbb{F}^m$ is called a **Finsler manifold of scalar curvature** $K(x, y)$. If moreover $K(x, y)$ is a constant k on TM°, then $\mathbb{F}^m$ is said to be a **Finsler manifold of constant curvature** k. Thus, by using (3.50) and (3.71) into (3.84), we deduce that $\mathbb{F}^m$ is of constant curvature k if and only if

$$R_{ab}(x, y) = kF^2(x, y)h_{ab}(x, y), \quad \forall\, (x, y) \in TM^\circ. \tag{3.85}$$

The next theorem will allow us to relate the geometry of foliations on TM° with the geometry of Finsler manifolds of positive constant curvature.

Theorem 3.22. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold and $k \in \mathbb{R}_+$. Then $\mathbb{F}^m$ is of positive constant curvature k if and only if the curvature–angular form Λ vanishes identically on the indicatrix bundle $IM(c)$, where $c = \dfrac{1}{\sqrt{k}}\cdot$*

Proof. Suppose $\mathbb{F}^m$ is a Finsler manifold of positive constant curvature k. The indicatrix bundle $IM(c)$ with $c = \dfrac{1}{\sqrt{k}}$ is non–empty and has the equation $F(x, y) = \dfrac{1}{\sqrt{k}}\cdot$ Then from (3.85) we obtain

$$R_{ab}(x, y) = h_{ab}(x, y), \quad \forall\, (x, y) \in IM(c). \tag{3.86}$$

Thus, according to (3.74), we have $\Lambda_{ab}(x, y) = 0$ for any $(x, y) \in IM(c)$. Hence the curvature–angular form Λ vanishes on $IM(c)$. Conversely, suppose $\Lambda = 0$ on $IM(c)$, that is, we have (3.86). Thus (3.85), which we want to prove, is true for any $(x, y) \in IM(c)$. Now, take a point $(x, y) \in TM^\circ \backslash IM(c)$. Since TM° admits the fundamental foliation $\mathcal{F}_F$, there exists $c^* > 0$ such that $(x, y) \in IM(c^*)$, and hence $F(x, y) = c^*$. Since F is positively homogeneous of degree 1, we deduce that $F\left(x, \dfrac{c}{c^*}\, y\right) = c$, which means that $\left(x, \dfrac{c}{c^*}\, y\right) \in IM(c)$. Thus, from (3.86), we obtain

$$R_{ab}\left(x, \frac{c}{c^*}\, y\right) = h_{ab}\left(x, \frac{c}{c^*}\, y\right)\cdot \tag{3.87}$$

Now, from (3.72) it follows that h_{ab} are positively homogeneous of degree zero with respect to (y^a). On the other hand, taking into account that G_a^b are positively homogeneous of degree 1 (see (3.7) and (3.5)), from (3.12) and (3.71) we deduce that R_{ab} are positively homogeneous of degree 2 with respect to (y^a). Hence form (3.87) we obtain

$$R_{ab}(x, y) = \frac{(c^*)^2}{c^2}\, h_{ab}(x, y) = kF^2(x, y)h_{ab}(x, y),\ \forall\, (x, y) \in IM(c^*),$$

that is, (3.85) is satisfied at any point of $IM(c^*)$. Thus (3.85) is true at any point of TM°, and therefore $\mathbb{F}^m$ is a Finsler manifold of positive constant curvature k. ∎

Finally, we combine Theorems 3.20, 3.21 and 3.22 and obtain the following interesting characterizations of Finsler manifolds of positive constant curvature.

Theorem 3.23. *Let* $\mathbb{F}^m = (M, F)$ *be a Finsler manifold and* k, c *two positive numbers such that* $c\sqrt{k} = 1$. *Then the following assertions are equivalent:*

(i) $\mathbb{F}^m$ *is a Finsler manifold of constant curvature* k.
(ii) *The Sasaki–Finsler metric* G *on the indicatrix bundle* $IM(c)$ *is bundle–like for the foliation determined by the transversal Liouville vector field* L^* *on* $IM(c)$.
(iii) *The transversal Liouville vector field is a Killing vector field on* $(IM(c), G)$.
(iv) *The curvature–angular form* Λ *vanishes identically on* $IM(c)$.

The equivalence of (i) and (iii) was proved by Bejancu and Farran [BF00b] for $k = 1$. If, in particular, $\mathbb{F}^m = (M, F)$ is a Riemannian manifold, then $IM(1)$ is known as the **tangent sphere bundle**. Tashiro [Tash69] investigated the geometry of a Riemannian manifold by using a contact metric structure on the tangent sphere bundle. The above results establish a new approach for studying the geometry of Finsler (Riemannian) manifolds. This is done by investigating the geometry of the natural foliations induced on the tangent bundles of such manifolds. The next theorem is another step in this direction.

Let (M, g) be a Riemannian manifold and $\{{}_b{}^a{}_c\}$ be the Christoffel coefficients (see 1.5.12)). In this case, the canonical transversal distribution HTM is defined by (3.6) where G_a^b are given by

$$G_a^b(x, y) = y^c \left\{ \begin{matrix} & b & \\ c & & a \end{matrix} \right\} (x).$$

Then it is easy to check that the curvature tensor field $R' = (R'_a{}^b{}_{cd})$ of the Levi–Civita connection ∇' on (M, g) satisfies the identities

$$y^a R'_a{}^b{}_{cd}(x) = R^b{}_{cd}(x, y) \quad \text{and} \quad \frac{\partial R^b{}_{cd}}{\partial y^a}(x, y) = R'_a{}^b{}_{cd}(x).$$

Finally, recall that when $R' = 0$ on M we say that (M, g) is locally Euclidean. When a Finsler manifold is Riemannian with locally Euclidean metric, we say that the Finsler metric given by (3.4) is **locally Euclidean**. Now we prove the following.

Theorem 3.24. *Let* $\mathbb{F}^m = (M, F)$ *be a Finsler manifold. Then the following assertions are equivalent:*

(i) *The Finsler metric is locally Euclidean.*
(ii) *The Sasaki–Finsler metric* G *is bundle–like for the vertical foliation* $\mathcal{F}_V$ *and* HTM° *is integrable.*

(iii) *HTM° is an integrable distribution that is tangent to a totally geodesic foliation $\mathcal{F}_H$ on (TM°, G).*
(iv) *HTM° is parallel with respect to the Levi–Civita connection $\widetilde{\nabla}$ on (TM°, G).*
(v) *HTM° and VTM° are tangent distributions to two totally geodesic foliations on (TM°, G).*
(vi) *The Vrănceanu and Schouten–Van Kampen connections induced by $\widetilde{\nabla}$ on (TM°, G) with respect to (3.8) coincide.*

Proof. First, we note that the Finsler metric given by (3.4) is locally Euclidean if and only if

$$\text{(a) } C_a{}^c{}_b = 0 \quad \text{and} \quad \text{(b) } R^c{}_{ab} = 0. \tag{3.88}$$

Then, by using (3.11b), (3.88) and Theorem 3.8, we deduce that (i) and (ii) are equivalent. Now, suppose (3.88) is true. Then from (3.11b) and (3.19a) we infer that HTM° is integrable and the foliation $\mathcal{F}_H$ is totally geodesic. Conversely, we suppose (iii) is true and by using (3.11b) and (3.19a) we obtain (3.88). This proves the equivalence of (i) and (iii). Next, if (3.88) is true, then from (3.19a) and (3.19c) we have

$$\begin{aligned} &\text{(a)} \quad \widetilde{\nabla}_{\frac{\delta}{\delta x^b}} \frac{\delta}{\delta x^a} = F_a{}^c{}_b \frac{\delta}{\delta x^c}, \\ &\text{(b)} \quad \widetilde{\nabla}_{\frac{\partial}{\partial y^a}} \frac{\delta}{\delta x^b} = (F_a{}^c{}_b - G_a{}^c{}_b) \frac{\partial}{\partial y^b} \cdot \end{aligned} \tag{3.89}$$

But from (3.88a) it follows that $\mathbb{F}^m$ is Riemannian, so it is Landsberg. Then, taking into account (3.26), from (3.89b) we obtain

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^a}} \frac{\delta}{\delta x^b} = 0. \tag{3.90}$$

Thus, from (3.89a) and (3.90), we deduce that HTM° is parallel with respect to $\widetilde{\nabla}$. Hence (i)$\Longrightarrow$(iv). Now, suppose that (iv) is true. Since $\widetilde{\nabla}$ is torsion–free, by Proposition 4.1.4 we infer that HTM° is integrable. Hence, by (3.11b), we have (3.88b). Then, by using (3.88b) in (3.19a) and taking into account that HTM° is parallel with respect to $\widetilde{\nabla}$, we obtain (3.88a). Hence (iv)$\Longrightarrow$(i), proving the equivalence of (i) and (iv). Next, the equivalence of (iv) and (v) follows from Theorem 4.4.3. Finally, (v) is equivalent to (vi) via Theorem 1.5.8. ∎

5.4 Foliations on *CR*-Submanifolds

Many well known concepts for surfaces in $\mathbb{R}^3$ have been generalized to give corresponding concepts for submanifolds in general. One such generalization that concerns foliations and will be used in this section is that of a ruled surface of $\mathbb{R}^3$. We define it as follows. Let M be a submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then we say that M is a **ruled submanifold** if it carries a foliation whose leaves (**rulings**) are totally geodesic immersed in $(\widetilde{M}, \widetilde{g})$. Two chapters of the book by Rovenskii [Rov98] were dedicated to the theory of ruled submanifolds. We use here this theory to characterize some classes of CR–submanifolds.

When the ambient manifold has some additional geometric structures, the study of foliations on a submanifold should focus on interrelations between these structures and foliations. It is the purpose of this section to present a study of the geometry of CR–submanifolds (see Example 2.1.8) stressing on the links between the foliations on these submanifolds and the complex structure on the embedding manifold.

Let $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ be a Kähler manifold where $\widetilde{g}$ is the Riemannian metric and $\widetilde{J}$ is the complex structure on $\widetilde{M}$. Suppose M is a CR–submanifold of $\widetilde{M}$, that is, M admits two complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that

(i) $\mathcal{D}$ is $\widetilde{J}$–invariant, i.e., $\widetilde{J}(\mathcal{D}) = \mathcal{D}$.
(ii) $\mathcal{D}^\perp$ is $\widetilde{J}$–anti–invariant, i.e., $\widetilde{J}(\mathcal{D}^\perp) \subset TM^\perp$.

By Proposition 2.1.11 we know that any real hypersurface of $\widetilde{M}$ is an example of a CR–submanifold with $\mathcal{D} \neq \{0\}$ and $\mathcal{D}^\perp \neq \{0\}$. On the other hand, Theorem 2.1.12 states that $\mathcal{D}^\perp$ is always integrable , and therefore any CR–submanifold admits a $\widetilde{J}$–**anti–invariant** (**totally real**) **foliation** which we denote by $\mathcal{F}^\perp$.

To continue the study of the geometry of M we recall some concepts and facts from the general theory of submanifolds (see Chen [C73]). Let $\widetilde{\nabla}$ be the Levi–Civita connection defined by $\widetilde{g}$ on $\widetilde{M}$. Then we put

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{4.1}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \quad \forall X \in \Gamma(TM),\ N \in \Gamma(TM^\perp). \tag{4.2}$$

Here ∇ is the Levi–Civita connection on (M, g), where g is the induced Riemannian metric by $\widetilde{g}$ on M. Also, $\nabla^\perp$ is a linear connection on the normal bundle $TM^\perp$, which is called the **normal connection**. Finally, B and A_N are the **second fundamental form** of M and the **shape operator** of M associated to the normal section N of M respectively. It is important to note that B is a symmetric $F(M)$–bilinear form and A_N is a self–adjoint operator, that is, we have

$$\begin{aligned} &\text{(a)}\ \ B(X, Y) = B(Y, X) \quad \text{and} \\ &\text{(b)}\ \ g(A_N X, Y) = g(X, A_N Y), \end{aligned} \tag{4.3}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$. Moreover, B and A_N are related by

$$\widetilde{g}(B(X,Y),N) = g(A_N X, Y). \tag{4.4}$$

From Section 1.5 we recall that M is totally geodesic when B vanishes identically on M. Thus by (4.4) we deduce that M is totally geodesic if and only if $A_N = 0$ for any $N \in \Gamma(TM^{\perp})$.

Finally, denote by $\widetilde{R}$ and R the curvature tensor fields of $\widetilde{\nabla}$ and ∇ respectively. Then the **Gauss equation** for the immersion of (M, g) in $(\widetilde{M}, \widetilde{g})$ is written as follows:

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z,U) &= g(R(X,Y)Z,U) \\ &\quad + \widetilde{g}(B(X,Z),B(Y,U)) \\ &\quad - \widetilde{g}(B(Y,Z),B(X,U)), \end{aligned} \tag{4.5}$$

for any $X, Y, Z, U \in \Gamma(TM)$.

Now, taking into account the concepts we introduced for foliations we may say that $\mathcal{F}^{\perp}$ is a foliation on M with structural distribution $\mathcal{D}^{\perp}$ and transversal distribution $\mathcal{D}$. Then we denote by $h^{\perp}$ and h the second fundamental forms of $\mathcal{F}^{\perp}$ and $\mathcal{D}$ respectively (see Section 3.2). By the definition of a CR–submanifold we have the orthogonal decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}. \tag{4.6}$$

Also, the normal bundle $TM^{\perp}$ has the orthogonal decomposition

$$TM^{\perp} = \widetilde{J}(\mathcal{D}^{\perp}) \oplus \mu, \tag{4.7}$$

where μ is the complementary orthogonal vector bundle to $\widetilde{J}(\mathcal{D}^{\perp})$ in $TM^{\perp}$. We say that $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) is A_N–**invariant**, if $A_N X \in \Gamma(\mathcal{D})$ (resp. $A_N Z \in \Gamma(\mathcal{D}^{\perp})$) for any $X \in \Gamma(\mathcal{D})$ (resp. $Z \in \Gamma(\mathcal{D}^{\perp})$). Then we can state the following characterizations of totally geodesic $\widetilde{J}$–anti–invariant foliations on CR–submanifolds.

Theorem 4.1. *Let $\mathcal{F}^{\perp}$ be the $\widetilde{J}$–anti–invariant foliation on a CR–submanifold M of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then the following assertions are equivalent.*

(i) *$\mathcal{F}^{\perp}$ is totally geodesic.*
(ii) *The second fundamental form of M satisfies*

$$B(X,Y) \in \Gamma(\mu), \quad \forall\, X \in \Gamma(\mathcal{D}^{\perp}),\ Y \in \Gamma(\mathcal{D}). \tag{4.8}$$

(iii) *$\mathcal{D}^{\perp}$ is A_N–invariant for any $N \in \Gamma(\widetilde{J}\mathcal{D}^{\perp})$.*
(iv) *$\mathcal{D}$ is A_N–invariant for any $N \in \Gamma(\widetilde{J}\mathcal{D}^{\perp})$.*

Proof. First, by using (2.1.27b), (2.1.29), (4.2) and (4.4) we obtain

$$
\begin{aligned}
\widetilde{g}(\widetilde{J}(\nabla_X Z), Y) &= -\widetilde{g}(\nabla_X Z, \widetilde{J}Y) \\
&= -\widetilde{g}(\widetilde{\nabla}_X Z, \widetilde{J}Y) \\
&= \widetilde{g}(\widetilde{\nabla}_X \widetilde{J}Z, Y) \\
&= -g(A_{\widetilde{J}Z}X, Y) \\
&= -\widetilde{g}(B(X,Y), \widetilde{J}Z),
\end{aligned}
\tag{4.9}
$$

for any $X, Z \in \Gamma(\mathcal{D}^{\perp})$ and $Y \in \Gamma(\mathcal{D})$. Now, suppose that $\mathcal{F}^{\perp}$ is totally geodesic. Then, by (3.4.1) we have $h^{\perp} = 0$, or equivalently

$$
\nabla_X Z \in \Gamma(\mathcal{D}^{\perp}), \quad \forall\, X, Z \in \Gamma(\mathcal{D}^{\perp}). \tag{4.10}
$$

Then (4.8) follows from (4.9) by using (4.10) and (4.7). Conversely, if (4.8) is satisfied, then from (4.9) we deduce that $\widetilde{J}(\nabla_X Z)$ is orthogonal to $\mathcal{D}$. If P and Q are the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively, then we have $\widetilde{J}P(\nabla_X Z) = 0$. As $\widetilde{J}$ is an automorphism on $\Gamma(\mathcal{D})$ we conclude that $P(\nabla_X Z) = 0$, that is, $\nabla_X Z \in \Gamma(\mathcal{D}^{\perp})$. Hence $\mathcal{F}^{\perp}$ is totally geodesic. This proves the equivalence of (i) and (ii). Due to (4.4), we obtain the equivalence of (ii) and (iii). Finally, by (4.3b) we deduce that (iii) and (iv) are equivalent. ∎

We say that M is **mixed geodesic** if we have

$$
B(X, Y) = 0, \quad \forall\, X \in \Gamma(\mathcal{D}^{\perp}),\ Y \in \Gamma(\mathcal{D}). \tag{4.11}
$$

Also, when $\mu = \{0\}$ we say that a CR–submanifold is an **anti–holomorphic submanifold**. Thus, any real hypersurface of M is anti–holomorphic. Now, from Theorem 4.1 we have the following corollaries.

Corollary 4.2. *Let M be a mixed geodesic CR–submanifold of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then the $\widetilde{J}$–anti–invariant foliation is totally geodesic.*

Corollary 4.3. *If M is a totally geodesic CR–submanifold of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$, then the $\widetilde{J}$–anti–invariant foliation is totally geodesic.*

Corollary 4.4. *Let M be an anti–holomorphic submanifold of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then M is mixed geodesic if and only if the $\widetilde{J}$–anti–invariant foliation is totally geodesic.*

Corollary 4.4 gives us an interesting geometric characterization of mixed geodesic anti–holomorphic submanifolds. Namely, M is mixed geodesic if and only if any geodesic of a leaf of $\mathcal{D}^{\perp}$ is a geodesic of (M, g). Also, according to Corollary 4.3, when M is totally geodesic, then any geodesic of a leaf of $\mathcal{F}^{\perp}$ is a geodesic of M which in turn is a geodesic of $\widetilde{M}$. Thus, any geodesic of a leaf of $\mathcal{F}^{\perp}$ is a geodesic of $\widetilde{M}$, which means that any leaf of $\mathcal{F}^{\perp}$ is totally geodesic immersed in $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. A CR–submanifold which is a ruled submanifold with respect to the foliation $\mathcal{F}^{\perp}$ is called a **totally real ruled CR–submanifold**. Then the above discussion enables us to state the following.

Corollary 4.5. *Any totally geodesic CR–submanifold of a Kähler manifold is a totally real ruled CR–submanifold.*

Now, we can present characterizations of a totally real ruled CR–submanifold by means of the geometric objects induced on its normal bundle. For one of these characterizations we need the following definition. We say that the vector bundle $J\mathcal{D}^{\perp}$ is $\mathcal{D}^{\perp}$–**parallel** if we have

$$\nabla^{\perp}_X \widetilde{J}Z \in \Gamma(\widetilde{J}\mathcal{D}^{\perp}), \quad \forall\, X, Z \in \Gamma(\mathcal{D}^{\perp}). \tag{4.12}$$

Theorem 4.6. *Let M be a CR–submanifold of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then the following assertions are equivalent.*

(i) *M is a totally real ruled CR–submanifold.*
(ii) *The second fundamental form of M satisfies* (4.8) *and*

$$B(X, Z) = 0, \quad \forall\, X, Z \in \Gamma(\mathcal{D}^{\perp}). \tag{4.13}$$

(iii) *$\widetilde{J}(\mathcal{D}^{\perp})$ is $\mathcal{D}^{\perp}$–parallel and the second fundamental form of M satisfies*

$$B(X, Y) \in \Gamma(\mu), \quad \forall\, X \in \Gamma(\mathcal{D}^{\perp}),\ Y \in \Gamma(TM). \tag{4.14}$$

(iv) *The shape operators of M satisfy*

$$A_{\widetilde{J}Z} X = 0, \quad \forall\, X, Z \in \Gamma(\mathcal{D}^{\perp}), \tag{4.15}$$

and

$$A_N X \in \Gamma(\mathcal{D}), \quad \forall\, X \in \Gamma(\mathcal{D}^{\perp}) \ \ \textit{and} \ \ N \in \Gamma(\mu). \tag{4.16}$$

Proof. By using (4.1) and (3.2.8a) we obtain

$$\widetilde{\nabla}_X Z = \nabla^{\mathcal{D}^{\perp}}_X Z + h^{\perp}(X, Z) + B(X, Z), \quad \forall\, X, Z \in \Gamma(\mathcal{D}^{\perp}), \tag{4.17}$$

where $\nabla^{\mathcal{D}^{\perp}}$ is the induced connection by ∇ on $\mathcal{D}^{\perp}$ (see Section 3.2). As the last two terms in (4.17) belong to complementary vector bundles, we deduce that any leaf of $\mathcal{D}^{\perp}$ is totally geodesic immersed in $\widetilde{M}$ if and only if

$$\begin{aligned} &\text{(a)} \quad h^{\perp} = 0, \quad \text{and} \\ &\text{(b)} \quad B(X, Z) = 0, \quad \forall\, X, Z \in \Gamma(\mathcal{D}^{\perp}). \end{aligned} \tag{4.18}$$

By Theorem 4.1 we see that (4.18a) is equivalent to (4.8). Thus the equivalence of (i) and (ii) is proved. Next, by using (2.1.27a), (2.1.29), (4.1), (4.2) and (4.4) we obtain

$$\widetilde{g}(\widetilde{\nabla}_X Z, U) = \widetilde{g}(\widetilde{\nabla}_X \widetilde{J}Z, \widetilde{J}U) = -\widetilde{g}(B(X, \widetilde{J}U), \widetilde{J}Z), \tag{4.19}$$

$$\widetilde{g}(\widetilde{\nabla}_X Z, \widetilde{J}W) = \widetilde{g}(B(X, Z), \widetilde{J}W), \tag{4.20}$$

$$\widetilde{g}(\widetilde{\nabla}_X Z, N) = \widetilde{g}(\widetilde{\nabla}_X \widetilde{J}Z, \widetilde{J}N) = \widetilde{g}(\nabla^{\perp}_X \widetilde{J}Z, \widetilde{J}N), \tag{4.21}$$

for any $X, Z, W \in \Gamma(\mathcal{D}^{\perp})$, $U \in \Gamma(\mathcal{D})$ and $N \in \Gamma(\mu)$. As M is a totally real ruled CR–submanifold if and only if $\widetilde{\nabla}_X Z \in \Gamma(\mathcal{D}^{\perp})$ for any $X, Z \in \Gamma(\mathcal{D}^{\perp})$, from (4.19), (4.20) and (4.21) we deduce the equivalence of (i) and (iii). Finally, by using (4.4) it follows that (ii) and (iv) are equivalent. Thus the proof is complete. ∎

Now, we present the necessary and sufficient conditions under which the Riemannian metric g is bundle–like for the totally real foliation $\mathcal{F}^{\perp}$ on (M, g).

Theorem 4.7. *Let M be a CR–submanifold of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then the following assertions are equivalent:*

(i) *The induced Riemannian metric g on M is bundle–like for the totally real foliation $\mathcal{F}^{\perp}$.*
(ii) *The second fundamental form of M satisfies*

$$B(U, \widetilde{J}V) + B(V, \widetilde{J}U) \in \Gamma(\mu), \quad \forall\, U, V \in \Gamma(\mathcal{D}). \tag{4.22}$$

Proof. By using (3.3.7) we deduce that g is bundle–like for $\mathcal{F}^{\perp}$ if and only if the Levi–Civita connection ∇ on (M, g) satisfies

$$g(\nabla_U X, V) + g(\nabla_V X, U) = 0, \quad \forall\, X \in \Gamma(\mathcal{D}^{\perp}),\ U, V \in \Gamma(\mathcal{D}). \tag{4.23}$$

Then, by using (4.1), (2.1.27a) and (2.1.29), we see that (4.23) is equivalent to

$$\widetilde{g}(\widetilde{\nabla}_U \widetilde{J}X, \widetilde{J}V) + \widetilde{g}(\widetilde{\nabla}_V \widetilde{J}X, \widetilde{J}U) = 0. \tag{4.24}$$

Finally, by using (1.5.9) and (4.1) in (4.24), we obtain

$$g(\widetilde{J}X, B(U, \widetilde{J}V) + B(V, \widetilde{J}U)) = 0,$$

which proves the equivalence of the assertions. ∎

Corollary 4.8. *Let (M, g) be an anti–holomorphic submanifold of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then g is bundle–like for $\mathcal{F}^{\perp}$ if and only if*

$$B(U, \widetilde{J}V) + B(V, \widetilde{J}U) = 0, \quad \forall\, U, V \in \Gamma(\mathcal{D}). \tag{4.25}$$

By combining Corollaries 4.4 and 4.8 we obtain the following.

Corollary 4.9. *Let (M, g) be an anti–holomorphic submanifold of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then $\mathcal{F}^{\perp}$ is totally geodesic with bundle–like metric g if and only if* (4.11) *and* (4.25) *are satisfied.*

Next, we discuss the integrability of the $\widetilde{J}$–invariant distribution $\mathcal{D}$ and state some decomposition theorems for CR–submanifolds. First, we prove the following theorems.

Theorem 4.10. (Bejancu [B78]). *Let M be a CR–submanifold of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then the $\widetilde{J}$–invariant distribution $\mathcal{D}$ is integrable if and only if*

$$B(U, \widetilde{J}V) - B(V, \widetilde{J}U) \in \Gamma(\mu), \quad \forall\, U, V \in \Gamma(\mathcal{D}). \tag{4.26}$$

Proof. Let $U, V \in \Gamma(\mathcal{D})$. Then, by Frobenius theorem, $\mathcal{D}$ is integrable if and only if

$$g([U, V], X) = 0, \quad \forall\, X \in \Gamma(\mathcal{D}^{\perp}),$$

which is equivalent to

$$\widetilde{g}(\widetilde{J}\widetilde{\nabla}_U V - \widetilde{J}\widetilde{\nabla}_V U, \widetilde{J}X) = 0.$$

By using (2.1.29) and (4.1), we deduce that $\mathcal{D}$ is integrable if and only if

$$\widetilde{g}(B(U, \widetilde{J}V) - B(V, \widetilde{J}U), \widetilde{J}X) = 0, \quad \forall\, X \in \Gamma(\mathcal{D}^{\perp}).$$

This completes the proof of the theorem. ■

Theorem 4.11. (Chen [C81]). *Let M be a CR–submanifold of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then we have the following assertions:*

(i) *The $\widetilde{J}$–invariant distribution $\mathcal{D}$ is integrable and the foliation $\mathcal{F}$ defined by $\mathcal{D}$ is totally geodesic if and only if*

$$B(U, V) \in \Gamma(\mu), \quad \forall\, U, V \in \Gamma(\mathcal{D}). \tag{4.27}$$

(ii) *The $\widetilde{J}$–invariant distribution $\mathcal{D}$ is integrable and M is a ruled submanifold with respect to the foliation $\mathcal{F}$ determined by $\mathcal{D}$ if and only if*

$$B(U, V) = 0, \quad \forall\, U, V \in \Gamma(\mathcal{D}). \tag{4.28}$$

Proof. $\mathcal{D}$ is integrable and $\mathcal{F}$ is totally geodesic if and only if $\nabla_U W \in \Gamma(\mathcal{D})$ for any $U, W \in \Gamma(\mathcal{D})$, that is

$$g(\nabla_U W, X) = 0, \quad \forall\, X \in \Gamma(\mathcal{D}^{\perp}).$$

By (4.1) this is equivalent to

$$\widetilde{g}(\widetilde{\nabla}_U W, X) = 0, \quad \forall\, X \in \Gamma(\mathcal{D}^\perp).$$

Then, taking into account (2.1.27a) and (2.1.29), we write the above condition as follows

$$\widetilde{g}(\widetilde{\nabla}_U \widetilde{J}W, \widetilde{J}X) = 0, \quad \forall\, X \in \Gamma(\mathcal{D}^\perp),$$

which, by (4.1), is equivalent to

$$\widetilde{g}(B(U, \widetilde{J}W), \widetilde{J}X) = 0, \quad \forall\, X \in \Gamma(\mathcal{D}^\perp).$$

As $\widetilde{J}$ is an automorphism of $\mathcal{D}$ we conclude that $\mathcal{F}$ exists and it is totally geodesic if and only if (4.27) is satisfied. Next, we denote by $\nabla^{\mathcal{D}}$ the induced connection by ∇ on $\mathcal{D}$ and by h the second fundamental form of the distribution $\mathcal{D}$ (see (1.5.21)). Then, by (4.1) and (1.5.17), we have

$$\widetilde{\nabla}_U V = \nabla^{\mathcal{D}}_U V + h(U, V) + B(U, V), \quad \forall\, U, V \in \Gamma(\mathcal{D}). \tag{4.29}$$

Thus, $\mathcal{D}$ is integrable and its leaves are totally geodesic immersed in $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ if and only if $\widetilde{\nabla}_U V \in \Gamma(\mathcal{D})$ for any $U, V \in \Gamma(\mathcal{D})$. By (4.29) this is equivalent to

$$\begin{aligned} &\text{(a)} \;\; h(U, V) = 0 \quad \text{and} \\ &\text{(b)} \;\; B(U, V) = 0, \quad \forall\, U, V \in \Gamma(\mathcal{D}). \end{aligned} \tag{4.30}$$

But (4.30a) is the condition for $\mathcal{F}$ to be totally geodesic and hence it is equivalent to (4.27). As (4.30b) implies (4.27), the proof of (ii) is complete. ∎

Taking into account Corollary 4.5 and the assertion (ii) of Theorem 4.11, we state the following corollary.

Corollary 4.12. *Let M be a totally geodesic CR–submanifold of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then the $\widetilde{J}$–invariant distribution $\mathcal{D}$ is integrable and M is a ruled submanifold with respect to both foliations $\mathcal{F}$ and $\mathcal{F}^\perp$.*

Now, following Chen [C81], we say that a CR–submanifold of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is a CR–**product** if $\mathcal{D}$ is integrable and both foliations $\mathcal{F}$ and $\mathcal{F}^\perp$ determined by $\mathcal{D}$ and $\mathcal{D}^\perp$ are totally geodesic. Then, according to Theorems 4.4.3 and 4.4.2, both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are parallel with respect to the Levi–Civita connection ∇ on (M, g), and M is locally a Riemannian product $L \times L^\perp$ where L and $L^\perp$ are local leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively. From Corollary 4.12 we deduce the following.

Corollary 4.13. *Any totally geodesic CR–submanifold of a Kähler manifold is a CR–product.*

Moreover, from (i) of Theorem 4.11 and (ii) of Theorem 4.1, we deduce the following characterizations of a CR–product.

Theorem 4.14. *Let M be a CR–submanifold of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then M is a CR–product if and only if the second fundamental form of M satisfies*

$$B(X, U) \in \Gamma(\mu), \quad \forall\, X \in \Gamma(TM),\ U \in \Gamma(\mathcal{D}). \tag{4.31}$$

Corollary 4.15. *Let M be an anti–holomorphic submanifold of a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$. Then M is a CR–product if and only if*

$$B(X, U) = 0, \quad \forall\, X \in \Gamma(TM),\ U \in \Gamma(\mathcal{D}). \tag{4.32}$$

Next, let $\widetilde{M}(c)$ be a **complex space form**, that is $\widetilde{M}(c)$ is a Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ of constant holomorphic sectional curvature c. Then the curvature tensor field $\widetilde{R}$ of $\widetilde{M}(c)$ is given by

$$\begin{aligned} \widetilde{R}(X, Y)Z = \frac{c}{4}\Big\{&\widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y + \widetilde{g}(Z, \widetilde{J}Y)\widetilde{J}X \\ &- \widetilde{g}(Z, \widetilde{J}X)\widetilde{J}Y + 2\widetilde{g}(X, \widetilde{J}Y)\widetilde{J}Z\Big\}, \end{aligned} \tag{4.33}$$

for any $X, Y, Z \in \Gamma(T\widetilde{M}(c))$. If we denote by $R^{\mathcal{D}^\perp}$ the curvature tensor of the induced linear connection $\nabla^{\mathcal{D}^\perp}$ on $\mathcal{D}^\perp$ and by R the curvature tensor of the Levi–Civita connection ∇ on (M, g), then by (1.6.3) we have

$$\begin{aligned} g(R(X, Y)Z, Z') = {} & g(R^{\mathcal{D}^\perp}(X, Y)Z, Z') \\ & + g(h^\perp(X, Z), h^\perp(Y, Z')) \\ & - g(h^\perp(Y, Z), h^\perp(X, Z')), \end{aligned} \tag{4.34}$$

for any $X, Y \in \Gamma(TM)$ and $Z, Z' \in \Gamma(\mathcal{D}^\perp)$. Now, suppose that $\mathcal{F}^\perp$ is a totally geodesic foliation, that is $h^\perp = 0$ on $\Gamma(\mathcal{D}^\perp) \times \Gamma(\mathcal{D}^\perp)$. In this case, (4.34) and (4.5) imply

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X, Y)Z, Z') = {} & g(R^{\mathcal{D}^\perp}(X, Y)Z, Z') \\ & + g(B(X, Z), B(Y, Z')) \\ & - g(B(Y, Z), B(X, Z')), \end{aligned} \tag{4.35}$$

for any $X, Y, Z, Z' \in \Gamma(\mathcal{D}^\perp)$. If moreover, M is a totally real ruled CR–submanifold then, by (4.13), we see that (4.35) becomes

$$\widetilde{g}(\widetilde{R}(X, Y)Z, Z') = g(R^{\mathcal{D}^\perp}(X, Y)Z, Z'). \tag{4.36}$$

Theorem 4.16. *Let M be a totally real ruled CR–submanifold of a complex space form $\widetilde{M}(c)$. Then the totally real foliation $\mathcal{F}^\perp$ on M is of constant curvature $\frac{c}{4}$, that is,*

$$g(R^{\mathcal{D}^\perp}(X, Y)Y, X) = \frac{c}{4}, \tag{4.37}$$

for any two orthogonal unit vector fields $X, Y \in \Gamma(\mathcal{D}^{\perp})$.

Proof. It follows from (4.36), by using (4.33) and taking into account that $\widetilde{J}\mathcal{D}^{\perp}$ is a subbundle of $TM^{\perp}$. ∎

Now, if $\mathcal{D}$ is integrable and the foliation $\mathcal{F}$ determined by $\mathcal{D}$ is totally geodesic, then from (1.6.3) we deduce that

$$g(R(U,V)W,W') = g(R^{\mathcal{D}}(U,V)W,W'), \tag{4.38}$$

for any $U, V, W, W' \in \Gamma(\mathcal{D})$. If moreover, M is a ruled submanifold with respect to $\mathcal{F}$, then by using (4.38), (4.5) and (4.28) we deduce that

$$\widetilde{g}(\widetilde{R}(U,\widetilde{J}U)\widetilde{J}U,U) = g(R^{\mathcal{D}}(U,\widetilde{J}U)\widetilde{J}U,U),$$

for any $U \in \Gamma(\mathcal{D})$. If we take U as unit vector field, by using (4.33) we deduce that

$$g(R^{\mathcal{D}}(U,\widetilde{J}U)\widetilde{J}U,U) = c,$$

which means that any leaf of $\mathcal{D}$ is a complex space form of holomorphic constant curvature c. Summing up this discussion and taking into account Theorem 4.16 we can state the following.

Theorem 4.17. *Let M be a CR–submanifold of a complex space form $\widetilde{M}(c)$. If $\mathcal{D}$ is integrable and M is a ruled submanifold with respect to both foliations $\mathcal{F}$ and $\mathcal{F}^{\perp}$, then M is a CR–product. Moreover, locally M is a Riemannian product $L \times L^{\perp}$ where L is a complex space form of constant holomorphic curvature c, and $L^{\perp}$ is a real space form of constant curvature $\dfrac{c}{4}\cdot$*

The concept of CR–submanifold of a Kähler manifold has been extended to submanifolds of manifolds endowed with various geometric structures like: locally conformal symplectic structures, contact metric structures, quaternionic structures, etc. It could be both interesting and useful to extend the study from this section to these structures.

6

A GAUGE THEORY ON A VECTOR BUNDLE

As it is well known, gauge theory has started as a mathematical formalism to provide a unified mathematical framework to describe the quantum field theories of electromagnetism, the weak interactions and the strong interactions. The original challenge was (still is) for a framework that unifies these with gravity as well.

Classically, gauge theories, used to deal with physical fields that live on a 4–dimensional Lorentz manifold (space time). The purpose of this chapter is to present a generalization of classical gauge theory. More precisely, we construct a gauge theory with respect to some physical fields defined on the total space E of a vector bundle $\xi = (E, \pi, M)$ over a smooth manifold M as a base space. But, total spaces of vector bundles admit a natural foliation, namely the foliation by fibers. Thus the theory presented in Sections 2.2, 2.3 and 2.4 to develop a tensor calculus on foliated manifolds can be used to investigate physical fields on such total spaces. So the physical fields Q^A, $A \in \{1, ..., q\}$, in this case will be expressed locally as $Q^A(x^1, ..., x^p; t^1, ..., t^n)$ where $(x^1, .., x^p)$ are the local coordinates of a point $x \in M$, and the perturbation parameters $(t^1, ..., t^n)$ represent the local coordinates of a point in the fiber $E_x = \pi^{-1}(x)$. In the first section we apply this tensor calculus theory to the particular case of the total space of a vector bundle. Then we study the global gauge invariance of Lagrangians on a vector bundle. In Section 6.3 we define the horizontal and vertical gauge covariant derivatives and give a method to obtain a local gauge invariant Lagrangian from a global gauge invariant Lagrangian. Also, we construct the horizontal, mixed and vertical Lagrangians for gauge fields and show that they are locally gauge invariant. In the last two sections we will display the deep involvement of the Vrănceanu connection into this study. By using it we obtain the equations of motion and the conservation laws for the full Lagrangian of the gauge theory on a vector bundle. Also, we derive the Bianchi identities for the strength fields of gauge fields. More about direction–dependent gauge theories can be found in Bejancu [B88], [B89].

The gauge theory we develop in this chapter suggests that some physical theories can be reconsidered to deal with a gauge theory that involves physical fields and Lagrangians that are functions which depend on more coordinates than the space time coordinates. This happens, for example, in the theory of supergravity as a generalization of the theory of gravity. This new theory uses two families of coordinates: the Bose coordinates and the Fermi coordinates.

6.1 Adapted Tensor Fields on the Total Space of a Vector Bundle

Let $\xi = (E, \pi, M)$ be a vector bundle with M as a base space, E as the total space and $\pi : E \to M$ as the projection mapping. Suppose M is a p–dimensional manifold and ξ is of rank n, that is, the fibers $E_x = \pi^{-1}(x)$ are n–dimensional for any $x \in M$. We choose the coordinates (x^α, t^i), $\alpha \in \{1, ..., p\}$, $i \in \{1, ..., n\}$, where (x^α) are the local coordinates on M. Then the transformation of coordinates on E is given by

$$\text{(a) } \widetilde{x}^\alpha = \widetilde{x}^\alpha(x^1, ..., x^p), \quad \text{(b) } \widetilde{t}^i = B^i_j(x^1, ..., x^p)t^j, \tag{1.1}$$

where B^i_j are real smooth functions locally defined on M and $\operatorname{rank}[B^i_j(x)] = n$ on any coordinate neighbourhood.

Throughout this chapter we shall use the following ranges for indices: $\alpha, \beta, \gamma, ... \in \{1, ..., p\}$; $i, j, k, ... \in \{1, ..., n\}$; $A, B, C, ... \in \{1, ..., q\}$ and $a, b, c, ... \in \{1, ..., r\}$.

From (1.1) it follows that

$$\text{(a) } \frac{\partial}{\partial t^j} = B^i_j(x)\frac{\partial}{\partial \widetilde{t}^i}, \text{ (b) } \frac{\partial}{\partial x^\alpha} = J^\beta_\alpha(x)\frac{\partial}{\partial \widetilde{x}^\beta} + \frac{\partial B^i_j}{\partial x^\alpha} t^j \frac{\partial}{\partial \widetilde{t}^i}, \tag{1.2}$$

where we put

$$J^\beta_\alpha(x) = \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha} \cdot \tag{1.3}$$

The tangent distribution to the foliation determined by the fibers of π is the vertical distribution on E and it is denoted by VE (see Example 2.1.4). Then $\left\{\frac{\partial}{\partial t^i}\right\}$, $i \in \{1, ..., n\}$, is a local basis for $\Gamma(VE)$. Next, suppose HE is a complementary distribution to VE in TE, that is, we have the decomposition

$$TE = VE \oplus HE. \tag{1.4}$$

We call HE the **horizontal distribution** on E. The existence of HE is guaranteed by the paracompactness of the manifold E. Now we apply the tensor calculus we developed in Section 2.2 to the particular case of the foliation determined by VE. Thus by (2.2.3) a local non–holonomic frame field on $\Gamma(HE)$ is $\left\{\frac{\delta}{\delta x^\alpha}\right\}$, $\alpha \in \{1, .., p\}$, given by

$$\frac{\delta}{\delta x^{\alpha}} = \frac{\partial}{\partial x^{\alpha}} - A^{i}_{\alpha}\,\frac{\partial}{\partial t^{i}}\,, \tag{1.5}$$

where A^i_α are np functions locally defined on E satisfying (see (2.2.5))

$$A^{j}_{\alpha}B^{i}_{j} = \widetilde{A}^{i}_{\beta}J^{\beta}_{\alpha} + \frac{\partial B^{i}_{j}}{\partial x^{\alpha}}\,t^{j}, \tag{1.6}$$

with respect to (1.1). Moreover, we have

$$\frac{\delta}{\delta x^{\alpha}} = J^{\beta}_{\alpha}\,\frac{\delta}{\delta \widetilde{x}^{\beta}}\,. \tag{1.7}$$

A smooth section of HE (resp. VE) is called a **horizontal** (resp. **vertical**) **vector field** on E. Similarly, a smooth section of the dual vector bundle HE^* (resp. VE^*) is called a **horizontal** (resp. **vertical**) 1–**form** on E. More generally, an **adapted tensor field** of type $(m, s; \ell, t)$ on E is an $F(E) - (m + \ell + s + t)$–multilinear mapping

$$T : (\Gamma(VE^*))^m \times (\Gamma(HE^*))^{\ell} \times (\Gamma(VE))^s \times (\Gamma(HE))^t \longrightarrow F(E).$$

By using another approach, Miron [Mir82] introduced such geometric objects in order to develop a Finsler geometry on a vector bundle.

Locally, a horizontal vector field X and a vertical vector field Y on E are expressed as follows

$$\text{(a) } X = X^{\alpha}(x,t)\,\frac{\delta}{\delta x^{\alpha}} \quad \text{and} \quad \text{(b) } Y = Y^{i}(x,t)\,\frac{\partial}{\partial t^{i}}\,, \tag{1.8}$$

where X^α and Y^i satisfy

$$\widetilde{X}^{\beta} = J^{\beta}_{\alpha}X^{\alpha} \quad \text{and} \quad \text{(b) } \widetilde{Y}^{j} = B^{j}_{i}Y^{i}. \tag{1.9}$$

Now, we denote by $\{\delta t^i, dx^\alpha\}$ the dual semi–holonomic frame field of $\left\{\frac{\partial}{\partial t^i}, \frac{\delta}{\delta x^\alpha}\right\}$, where we put

$$\delta t^{i} = dt^{i} + A^{i}_{\alpha}dx^{\alpha}. \tag{1.10}$$

Then we have (see (2.2.12))

$$\text{(a) } \delta\widetilde{t^{i}} = B^{i}_{j}\delta t^{j} \quad \text{and} \quad \text{(b) } d\widetilde{x}^{\beta} = J^{\beta}_{\alpha}dx^{\alpha}, \tag{1.11}$$

with respect to (1.1). Thus a horizontal 1–form ω and a vertical 1–form Ω are locally expressed as follows:

$$\text{(a) } \omega = \omega_{\alpha}dx^{\alpha} \quad \text{and} \quad \text{(b) } \Omega = \Omega_{i}\delta t^{i}, \tag{1.12}$$

where ω_α and Ω_i satisfy

$$\text{(a) } \omega_\alpha = J^\beta_\alpha \widetilde{\omega}_\beta \quad \text{and} \quad \text{(b) } \Omega_j = B^i_j \widetilde{\Omega}_i. \tag{1.13}$$

In general, an adapted tensor field T of type $(m, s; \ell, t)$ is locally represented by $n^{m+s}p^{\ell+t}$ functions $T^{i_1\dots i_m\alpha_1\dots\alpha_\ell}_{j_1\dots j_s\beta_1\dots\beta_t}$ satisfying

$$\begin{aligned} &\widetilde{T}^{k_1\dots k_m\gamma_1\dots\gamma_\ell}_{h_1\dots h_s\varepsilon_1\dots\varepsilon_t} B^{h_1}_{j_1}\cdots B^{h_s}_{j_s} J^{\varepsilon_1}_{\beta_1}\cdots J^{\varepsilon_t}_{\beta_t} \\ &\qquad = T^{i_1\dots i_m\alpha_1\dots\alpha_\ell}_{j_1\dots j_s\beta_1\dots\beta_t} B^{k_1}_{i_1}\cdots B^{k_m}_{i_m} J^{\gamma_1}_{\alpha_1}\cdots J^{\gamma_\ell}_{\alpha_\ell}, \end{aligned} \tag{1.14}$$

with respect to (1.1). Certainly, horizontal and vertical vector fields and 1–forms are examples of adapted tensor fields on E. Also, according to Lemma 2.2.4

$$T_\alpha{}^i{}_\beta = \frac{\delta A^i_\alpha}{\delta x^\beta} - \frac{\delta A^i_\beta}{\delta x^\alpha}\,, \quad \begin{array}{l} i \in \{1, ..., n\}, \\ \alpha, \beta \in \{1, ..., p\}, \end{array} \tag{1.15}$$

define an adapted tensor field on E of type $(1, 0; 0, 2)$. This is the integrability tensor of the horizontal distribution HE since we have (cf. (2.2.18))

$$\left[\frac{\delta}{\delta x^\alpha}\,, \frac{\delta}{\delta x^\beta}\right] = T_\alpha{}^i{}_\beta \,\frac{\partial}{\partial t^i}\,\cdot \tag{1.16}$$

Next, let ∇ be an adapted linear connection on E, that is, we have (cf. (1.2.1) and (1.2.2))

$$\text{(a) } \nabla_Z X \in \Gamma(HE) \quad \text{and} \quad \text{(b) } \nabla_Z Y \in \Gamma(VE), \tag{1.17}$$

for any $X \in \Gamma(HE)$, $Y \in \Gamma(VE)$ and $Z \in \Gamma(TM)$. Then we put

$$\text{(a) } \nabla_{\frac{\delta}{\delta x^\beta}} \frac{\delta}{\delta x^\alpha} = F_\alpha{}^\gamma{}_\beta \,\frac{\delta}{\delta x^\gamma}\,, \quad \text{(b) } \nabla_{\frac{\partial}{\partial t^i}} \frac{\delta}{\delta x^\alpha} = L_\alpha{}^\gamma{}_i \,\frac{\delta}{\delta x^\gamma}\,, \tag{1.18}$$

and

$$\text{(a) } \nabla_{\frac{\delta}{\delta x^\beta}} \frac{\partial}{\partial t^i} = D_i{}^k{}_\beta \,\frac{\partial}{\partial t^k}\,, \quad \text{(b) } \nabla_{\frac{\partial}{\partial t^j}} \frac{\partial}{\partial t^i} = C_i{}^k{}_j \,\frac{\partial}{\partial t^k}\,\cdot \tag{1.19}$$

As we know from Section 2.3, the adapted linear connection ∇ defines two types of covariant derivatives for adapted tensor fields: the transversal and structural covariant derivatives. Here, according to the names of VE and HE, we call them the horizontal and vertical covariant derivatives. Thus, if X is a horizontal vector field given by (1.8a), then its **horizontal** and **vertical covariant derivatives** are given by

$$X^\alpha{}_{|\beta} = \frac{\delta X^\alpha}{\delta x^\beta} + X^\gamma F_\gamma{}^\alpha{}_\beta, \tag{1.20}$$

and

$$X^\alpha{}_{\|i} = \frac{\partial X^\alpha}{\partial t^i} + X^\gamma L_\gamma{}^\alpha{}_i, \tag{1.21}$$

respectively. Similarly, the **horizontal** and **vertical covariant derivatives** of a vertical vector field Y given by (1.8b) are given by

$$Y^i{}_{|\alpha} = \frac{\delta Y^i}{\delta x^\alpha} + Y^j D_j{}^i{}_\alpha, \tag{1.22}$$

and

$$Y^i{}_{\|j} = \frac{\partial Y^i}{\partial t^j} + Y^k C_k{}^i{}_j, \tag{1.23}$$

respectively. For the horizontal 1–form ω given by (1.12a) we have

$$\text{(a) } \omega_{\alpha|\beta} = \frac{\delta \omega_\alpha}{\delta x^\beta} - \omega_\gamma F_\alpha{}^\gamma{}_\beta, \quad \text{(b) } \omega_{\alpha\|i} = \frac{\partial \omega_\alpha}{\partial t^i} - \omega_\gamma L_\alpha{}^\gamma{}_i. \tag{1.24}$$

Similarly, for the vertical 1–form Ω from (1.12b) we obtain

$$\text{(a) } \Omega_{i|\alpha} = \frac{\delta \Omega_i}{\delta x^\alpha} - \Omega_k D_i{}^k{}_\alpha, \quad \text{(b) } \Omega_{i\|j} = \frac{\partial \Omega_i}{\partial t^j} - \Omega_k C_i{}^k{}_j. \tag{1.25}$$

The general formulas (2.3.17) and (2.3.18) for transversal and structural covariant derivatives of an adapted tensor field on a foliated manifold give us the corresponding formulas for horizontal and vertical covariant derivatives on E. We only write them for an adapted tensor field T with local components $T^{i\alpha}_{j\beta}$. Thus, applying (2.3.17) and using (1.18) and (1.19), we obtain for the **horizontal covariant derivative** of T the following formula

$$T^{i\alpha}_{j\beta|\gamma} = \frac{\delta T^{i\alpha}_{j\beta}}{\delta x^\gamma} + T^{k\alpha}_{j\beta} D_k{}^i{}_\gamma + T^{i\varepsilon}_{j\beta} F_\varepsilon{}^\alpha{}_\gamma - T^{i\alpha}_{k\beta} D_j{}^k{}_\gamma - T^{i\alpha}_{j\varepsilon} F_\beta{}^\varepsilon{}_\gamma. \tag{1.26}$$

Similarly, the **vertical covariant derivative** of T is given by

$$T^{i\alpha}_{j\beta\|k} = \frac{\partial T^{i\alpha}_{j\beta}}{\partial t^k} + T^{h\alpha}_{j\beta} C_h{}^i{}_k + T^{i\varepsilon}_{j\beta} L_\varepsilon{}^\alpha{}_k - T^{i\alpha}_{h\beta} C_j{}^h{}_k - T^{i\alpha}_{j\varepsilon} L_\beta{}^\varepsilon{}_k. \tag{1.27}$$

The local components of the torsion tensor field of the adapted linear connection $\nabla = \{F_\alpha{}^\gamma{}_\beta, L_\alpha{}^\gamma{}_i, D_i{}^k{}_\alpha, C_i{}^k{}_j\}$ are given by $T_\alpha{}^i{}_\beta$ from (1.15) and

$$\begin{aligned} &\text{(a) } T_i{}^k{}_j = C_i{}^k{}_j - C_j{}^k{}_i, \text{ (b) } T_\alpha{}^k{}_j = \frac{\partial A^k_\alpha}{\partial t^j} - D_j{}^k{}_\alpha, \\ &\text{(c) } T_\alpha{}^\gamma{}_j = L_\alpha{}^\gamma{}_j, \qquad \text{(d) } T_\alpha{}^\gamma{}_\beta = F_\alpha{}^\gamma{}_\beta - F_\beta{}^\gamma{}_\alpha. \end{aligned} \tag{1.28}$$

Next, we suppose that the total space E of the vector bundle ξ is endowed with a semi–Riemannian metric g and the vertical distribution is also semi–Riemannian (non–degenerate) with respect to g. Then we choose the complementary orthogonal distribution to VE in TE as horizontal distribution HE. Thus HE is semi–Riemannian too. In this case the functions A^i_α from (1.5) are determined by g as follows (see (3.1.20))

$$A^i_\alpha = g^{ij} g_{j\alpha}, \tag{1.29}$$

where

$$g_{j\alpha} = g\left(\frac{\partial}{\partial t^j}, \frac{\partial}{\partial x^\alpha}\right),$$

and g^{ij} are the entries of the inverse matrix of $[g_{ij}]$, where

$$g_{ij} = g\left(\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}\right). \tag{1.30}$$

We also put

$$h_{\alpha\beta} = g\left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}\right). \tag{1.31}$$

Now, we consider the Levi–Civita connection ∇ on (E, g) and the Vrănceanu connection ∇^* with respect to ∇ (see Section 3.1), which is an adapted linear connection on E. Thus, ∇^* is given by (see (3.1.12))

$$\nabla^*_X Y = Q\nabla_{QX} QY + Q'\nabla_{Q'X} Q'Y + Q[Q'X, QY] + Q'[QX, Q'Y], \tag{1.32}$$

for any $X, Y \in \Gamma(TE)$, where Q and Q' are the projection morphisms of TE on VE and HE respectively. The local coefficients of ∇^* are given by (see Proposition 3.1.2)

$$\begin{aligned}
&\text{(a) } C_i{}^k{}_j = \frac{1}{2}\, g^{kh}\left(\frac{\partial g_{hi}}{\partial t^j} + \frac{\partial g_{hj}}{\partial t^i} - \frac{\partial g_{ij}}{\partial t^h}\right), \quad \text{(b) } D_i{}^k{}_\alpha = \frac{\partial A^k_\alpha}{\partial t^i},\\
&\text{(c) } L_\alpha{}^\gamma{}_i = 0, \quad \text{(d) } F_\alpha{}^\gamma{}_\beta = \frac{1}{2}\, h^{\beta\mu}\left(\frac{\delta h_{\mu\alpha}}{\delta x^\beta} + \frac{\delta h_{\mu\beta}}{\delta x^\alpha} - \frac{\delta h_{\alpha\beta}}{\delta x^\mu}\right),
\end{aligned} \tag{1.33}$$

where $h^{\beta\mu}$ are the entries of the inverse matrix of $[h_{\alpha\beta}]$. Thus, by (1.28) and (1.33), all the local components of the torsion tensor field of ∇^* vanish, except $T_\alpha{}^i{}_\beta$ given by (1.15). We note that $\{h_{\alpha\beta}\}$ and $\{g_{ij}\}$ are the local components of an adapted tensor field of type $(0, 0; 0, 2)$ and $(0, 2; 0, 0)$ respectively on E. Moreover, from Proposition 3.1.8 we deduce that the Vrănceanu connection ∇^* is h–**metrical** and v–**metrical**, that is, we have

$$\text{(a) } h_{\alpha\beta|\gamma} = 0 \quad \text{and} \quad \text{(b) } g_{ij\|k} = 0. \tag{1.34}$$

Finally, by (1.33b) and (2.3.21), we deduce that

$$\left[\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial t^i}\right] = \frac{\partial A^k_\alpha}{\partial t^i}\,\frac{\partial}{\partial t^k} = D_i{}^k{}_\alpha\,\frac{\partial}{\partial t^k}. \tag{1.35}$$

The above properties of the Vrănceanu connection will enable us to develop, in the remaining part of this chapter, a gauge theory on the total space of a vector bundle.

6.2 Global Gauge Invariance of Lagrangians on a Vector Bundle

Let $\xi = (E, \pi, M)$ be a vector bundle and $Q^A : M \to \mathbb{R}$, $A \in \{1, ..., q\}$, be some physical fields on the base manifold M. As it is well known (see Chaichian–Nelipa [CN84]), the simplest Lagrangian on M is of the following form

$$L_0(x) = L\left(Q^A(x), \frac{\partial Q^A}{\partial x^\alpha}(x)\right), \tag{2.1}$$

where L is a real smooth function on a domain of $\mathbb{R}^{q(1+p)}$.

Now, we consider some scalar fields $Q^A(x,t)$, $A \in \{1, ..., q\}$, on E. Then we note that by (1.7) the transformation of $\frac{\delta Q^A}{\delta x^\alpha}(x,t)$ with respect to (1.1) on E is the same as the transformation of $\frac{\partial Q^A}{\partial x^\alpha}(x)$ with respect to (1.1a) on M. This enables us to obtain from (2.1) a Lagrangian on E given by

$$L'_0(x,t) = L\left(Q^A(x,t), \frac{\delta Q^A}{\delta x^\alpha}(x,t)\right), \tag{2.2}$$

where, this time, $Q^A : E \to \mathbb{R}$. Thus we have a general method to construct Lagrangians on the total space of a vector bundle from Lagrangians on the base space, provided there exists on E a horizontal distribution HE. As (1.5) shows, the Lagrangian (2.2) contains both types of partial derivatives $\frac{\partial Q^A}{\partial x^\alpha}(x,t)$ and $\frac{\partial Q^A}{\partial t^i}(x,t)$ but incorporated in $\frac{\delta Q^A}{\delta x^\alpha}(x,t)$. Next, we suppose that E is endowed with a semi–Riemannian metric g such that VE is a non–degenerate distribution. Then we consider a Lagrangian on E of the following general form

$$\mathcal{L}_0(x,t) = \mathcal{L}\left(Q^A(x,t), \frac{\delta Q^A}{\delta x^\alpha}(x,t), \frac{\partial Q^A}{\partial t^i}(x,t)\right), \tag{2.3}$$

where $\mathcal{L}$ is a smooth function on a domain of $\mathbb{R}^s$, $s = q(1+p+n)$. As we have seen in Section 6.1, $h_{\alpha\beta}$ and g_{ij} determine some adapted tensor fields on E. Thus, according to (1.14), we have

$$h_{\alpha\beta}(x,t) = \widetilde{h}_{\gamma\mu}(\widetilde{x},\widetilde{t})J^\gamma_\alpha(x)J^\mu_\beta(x), \tag{2.4}$$

and

$$g_{ij}(x,t) = \widetilde{g}_{hk}(\widetilde{x},\widetilde{t})B^h_i(x)B^k_j(x), \tag{2.5}$$

with respect to the change of coordinates (1.1) on E. Also, it is easy to see from (1.7) and (1.2a) that $\frac{\delta Q^A}{\delta x^\alpha}$ and $\frac{\partial Q^A}{\partial t^i}$ are the local components of a horizontal and vertical 1–form respectively on E, for each $A \in \{1, ..., q\}$. Now, we define locally on E and M the functions:

$$\begin{aligned} &\text{(a)} \quad H(x,t) = (|\det[h_{\alpha\beta}(x,t)]|)^{1/2}, \\ &\text{(b)} \quad V(x,t) = (|\det[g_{ij}(x,t)]|)^{1/2}, \end{aligned} \tag{2.6}$$

and

$$\text{(a) } J(x) = \det[J^{\beta}_{\alpha}(x)], \quad \text{(b) } B(x) = \det[B^{j}_{i}(x)], \tag{2.7}$$

respectively. Then, by using (2.4)–(2.7), we obtain

$$\text{(a) } H(x,t) = \widetilde{H}(\widetilde{x},\widetilde{t})|J(x)| \quad \text{and} \quad \text{(b) } V(x,t) = \widetilde{V}(\widetilde{x},\widetilde{t})|B(x)|. \tag{2.8}$$

Further, we define locally the function

$$\mathcal{L}^*_0(x,t) = \mathcal{L}_0(x,t)H(x,t)V(x,t). \tag{2.9}$$

Then, by using (2.8) and (2.9), we deduce that

$$\mathcal{L}^*_0(x,t) = \mathcal{L}^*_0(\widetilde{x},\widetilde{t})J(x)B(x), \tag{2.10}$$

provided E is an orientable manifold. Thus $\mathcal{L}^*_0(x,t)$ is a Lagrangian density on E which enables us to define the functional

$$I(\Omega) = \int_{\Omega} \mathcal{L}^*_0(x,t)dx^1 \wedge \cdots \wedge dx^p \wedge dt^1 \wedge \cdots \wedge dt^n, \tag{2.11}$$

where Ω is a compact domain of E. By using (1.10) it is easy to see that

$$dx^1 \wedge \cdots \wedge dx^p \wedge dt^1 \wedge \cdots \wedge dt^n = dx^1 \wedge \cdots \wedge dx^p \wedge \delta t^1 \wedge \cdots \delta t^n,$$

which together with (2.10) implies that $I(\Omega)$ is independent of coordinates on E.

Next, the variational principle

$$\delta(I(\Omega)) = 0$$

implies the following **Euler–Lagrange equations** for $Q^A(x,t)$:

$$\frac{\partial \mathcal{L}^*_0}{\partial Q^A} - \frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial \mathcal{L}^*_0}{\partial\left(\dfrac{\partial Q^A}{\partial x^{\alpha}}\right)}\right) - \frac{\partial}{\partial t^i}\left(\frac{\partial \mathcal{L}^*_0}{\partial\left(\dfrac{\partial Q^A}{\partial t^i}\right)}\right) = 0. \tag{2.12}$$

In (2.12) and in some other lengthy formulas we omit the point (x,t) where the geometric objects are considered. Also, in (2.12) we have summations about both indices $\alpha \in \{1,...,p\}$ and $i \in \{1,...,n\}$. We want now to express (2.12) by using horizontal and vertical covariant derivatives with respect to the Vrănceanu connection. To this end we put

$$Q^{h\alpha}_{A} = \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\delta Q^A}{\delta x^{\alpha}}\right)}, \tag{2.13}$$

and

$$Q_A^{vi} = \frac{\partial \mathcal{L}}{\partial \left(\dfrac{\partial Q^A}{\partial t^i} \right)} \star \cdot \tag{2.14}$$

The star in (2.14) means that we take partial derivatives of $\mathcal{L}$ only with respect to variables $\dfrac{\partial Q^A}{\partial t^i}$ which do not appear in the expression of $\dfrac{\delta Q^A}{\delta x^\alpha} \cdot$ Then, by using (1.2a) and (1.7), we deduce that

$$\text{(a) } \widetilde{Q}_A^{h\beta} = Q_A^{h\alpha} J_\alpha^\beta(x) \quad \text{and} \quad \text{(b) } \widetilde{Q}_A^{vj} = Q_A^{vi} B_i^j(x). \tag{2.15}$$

Hence

$$\text{(a) } Q_A^h = Q_A^{h\alpha} \frac{\delta}{\delta x^\alpha} \quad \text{and} \quad \text{(b) } Q_A^v = Q_A^{vi} \frac{\partial}{\partial t^i}, \tag{2.16}$$

are q horizontal and vertical vector fields on E respectively. Now we can state the following.

Theorem 2.1. *The Euler–Lagrange equations for the scalar fields $Q^A(x,t)$, $A \in \{1, ..., q\}$, can be expressed in terms of the horizontal and vertical covariant derivatives induced by the Vrănceanu connection on E as follows*

$$\frac{\partial \mathcal{L}}{\partial Q^A} - Q_A^{h\alpha}{}_{|\alpha} - Q_A^{vi}{}_{\|i} = E_A, \tag{2.17}$$

where we put

$$\begin{aligned} E_A = &\left\{ \frac{1}{HV} \frac{\delta(HV)}{\delta x^\alpha} - D_i{}^i{}_\alpha - F_\alpha{}^\gamma{}_\gamma \right\} Q_A^{h\alpha} \\ &+ \left\{ \frac{1}{HV} \frac{\partial(HV)}{\partial t^j} - C_j{}^i{}_i \right\} Q_A^{vj}. \end{aligned} \tag{2.18}$$

Proof. First, by using (2.9) and (2.3), we obtain

$$\frac{\partial \mathcal{L}_0^*}{\partial Q^A} = \frac{\partial \mathcal{L}}{\partial Q^A} HV. \tag{2.19}$$

Next, by using (2.9), (1.5) and (2.13), we deduce that

$$\frac{\partial \mathcal{L}_0^*}{\partial \left(\dfrac{\partial Q^A}{\partial x^\alpha} \right)} = \frac{\partial \mathcal{L}}{\partial \left(\dfrac{\delta Q^A}{\delta x^\alpha} \right)} HV = Q_A^{h\alpha} HV. \tag{2.20}$$

Then, taking into account (1.5) and (1.20), from (2.20) we infer that

$$\begin{aligned}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}_0^*}{\partial\left(\dfrac{\partial Q^A}{\partial x^\alpha}\right)}\right) &= \left(\frac{\delta Q_A^{h\alpha}}{\delta x^\alpha} + A_\alpha^i \frac{\partial Q_A^{h\alpha}}{\partial t^i}\right) HV \\ &+ \left(\frac{\delta (HV)}{\delta x^\alpha} + A_\alpha^i \frac{\partial (HV)}{\partial t^i}\right) Q_A^{h\alpha} \\ &= \left(Q_{A\ |\alpha}^{h\alpha} - Q_A^{h\gamma} F_\gamma{}^\alpha{}_\alpha + A_\alpha^i \frac{\partial Q_A^{h\alpha}}{\partial t^i}\right) HV \\ &+ \left(\frac{\delta (HV)}{\delta x^\alpha} + A_\alpha^i \frac{\partial (HV)}{\partial t^i}\right) Q_A^{h\alpha}.\end{aligned} \tag{2.21}$$

In the next derivative we must be careful that $\dfrac{\partial Q^A}{\partial t^i}$ might appear in the expression of $\dfrac{\delta Q^A}{\delta x^\alpha}\cdot$ Thus, by (2.9) and (1.5), we obtain

$$\begin{aligned}\frac{\partial \mathcal{L}_0^*}{\partial\left(\dfrac{\partial Q^A}{\partial t^i}\right)} &= -\frac{\partial \mathcal{L}}{\partial\left(\dfrac{\delta Q^A}{\delta x^\alpha}\right)} A_\alpha^i HV + \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial Q^A}{\partial t^i}\right)} \star (HV) \\ &= -A_\alpha^i Q_A^{h\alpha} HV + Q_A^{vi} HV.\end{aligned}$$

Then take partial derivatives with respect to t^i and by using (1.22) and (1.33b) we deduce that

$$\begin{aligned}\frac{\partial}{\partial t^i}\left(\frac{\partial \mathcal{L}_0^*}{\partial\left(\dfrac{\partial Q^A}{\partial t^i}\right)}\right) &= -D_i{}^i{}_\alpha Q_A^{h\alpha} HV - A_\alpha^i \frac{\partial Q_A^{h\alpha}}{\partial t^i} HV \\ -A_\alpha^i Q_A^{h\alpha} \frac{\partial (HV)}{\partial t^i} &+ \left(Q_{A\ \|i}^{vi} + Q_A^{vj} C_j{}^i{}_i\right) HV + Q_A^{vi} \frac{\partial (HV)}{\partial t^i}\,.\end{aligned} \tag{2.22}$$

Finally, we use (2.19), (2.21) and (2.22) in (2.12) and taking into account (2.18) we obtain (2.17). ■

Next, we consider an r–dimensional Lie group G and denote by G^* its Lie algebra. Let V be a real q–dimensional vector space and $g\ell(V)$ be the Lie algebra of all endomorphisms of V with the bracket operation

$$[A, B] = AB - BA, \quad \forall\, A, B \in g\ell(V).$$

In what follows in this chapter we suppose that G^* has a q–**dimensional representation** ρ on V, that is, ρ is a homomorphism of Lie algebras of G^* into $g\ell(V)$. We fix a basis $\{X_a\}$, $a \in \{1, ..., r\}$, of the Lie algebra G^* and express any $X \in G^*$ by $X = \varepsilon^a X_a$, where ε^a, $a \in \{1, ..., r\}$, are real constants.

Now, a **global gauge action** of G on the physical fields $Q^A(x,t)$, $A \in \{1,...,q\}$, is given by the infinitesimal transformations

$$Q'^A(x,t) = Q^A(x,t) + \delta(Q^A(x,t)), \tag{2.23}$$

where we put

$$\delta(Q^A(x,t)) = \varepsilon^a [X_a]^A_B Q^B(x,t). \tag{2.24}$$

Here, by $[X_a]^A_B$ we denote the $q \times q$ matrix corresponding to X_a by the q–dimensional representation ρ. Applying the operators $\dfrac{\delta}{\delta x^\alpha}$ and $\dfrac{\partial}{\partial t^i}$ from the semi–holonomic frame field $\left\{ \dfrac{\partial}{\partial t^i}, \dfrac{\delta}{\delta x^\alpha} \right\}$ to (2.23) and taking into account (2.24), we obtain

$$\frac{\delta Q'^A}{\delta x^\alpha} = \frac{\delta Q^A}{\delta x^\alpha} + \delta\left(\frac{\delta Q^A}{\delta x^\alpha}\right), \tag{2.25}$$

and

$$\frac{\partial Q'^A}{\partial t^i} = \frac{\partial Q^A}{\partial t^i} + \delta\left(\frac{\partial Q^A}{\partial t^i}\right), \tag{2.26}$$

where we put

$$\delta\left(\frac{\delta Q^A}{\delta x^\alpha}\right) = \varepsilon^a [X_a]^A_B \frac{\delta Q^B}{\delta x^\alpha}, \tag{2.27}$$

and

$$\delta\left(\frac{\partial Q^A}{\partial t^i}\right) = \varepsilon^a [X_a]^A_B \frac{\partial Q^B}{\partial t^i}, \tag{2.28}$$

respectively. Next, we define locally the functions

$$J_a^{h\alpha} = -Q_A^{h\alpha} [X_a]^A_B Q^B, \tag{2.29}$$

and

$$J_a^{vi} = -Q_A^{vi} [X_a]^A_B Q^B. \tag{2.30}$$

As $Q_A^{h\alpha}$ and Q_A^{vi} are the local components of horizontal and vertical vector fields on E, we conclude that

$$\text{(a) } J_a^h = J_a^{h\alpha} \frac{\delta}{\delta x^\alpha} \quad \text{and} \quad \text{(b) } J_a^v = J_a^{vi} \frac{\partial}{\partial t^i}, \tag{2.31}$$

are horizontal and vertical vector fields on E respectively.

We call J_a^h and J_a^v, $a \in \{1,...,r\}$, the **horizontal** and **vertical currents** on E respectively. If $\mathcal{L}_0(x,t)$ from (2.3) is invariant with respect to the infinitesimal transformations (2.23), (2.25) and (2.26) we say that it is **globally gauge G–invariant**. Then we prove the following.

Proposition 2.2. *$\mathcal{L}_0(x,t)$ is globally gauge G–invariant if and only if for any $a \in \{1,...,r\}$ we have*

$$\left\{\frac{\partial \mathcal{L}}{\partial Q^A} Q^B + Q_A^{h\alpha} \frac{\delta Q^B}{\delta x^\alpha} + Q_A^{vi} \frac{\partial Q^B}{\partial t^i}\right\} [X_a]_B^A = 0. \tag{2.32}$$

Proof. $\mathcal{L}_0$ is globally gauge G–invariant if and only if $\delta\mathcal{L} = 0$, which is equivalent to

$$\frac{\partial \mathcal{L}}{\partial Q^A} \delta Q^A + \frac{\partial \mathcal{L}}{\partial \left(\frac{\delta Q^A}{\delta x^\alpha}\right)} \delta\left(\frac{\delta Q^A}{\delta x^\alpha}\right) + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial Q^A}{\partial t^i}\right)} \star \delta\left(\frac{\partial Q^A}{\partial t^i}\right) = 0. \tag{2.33}$$

Now, we use (2.24), (2.27), (2.28), (2.13) and (2.14) in (2.33) and obtain

$$\left\{\frac{\partial \mathcal{L}}{\partial Q^A} Q^B + Q_A^{h\alpha} \frac{\delta Q^B}{\delta x^\alpha} + Q_A^{vi} \frac{\partial Q^B}{\partial t^i}\right\} \varepsilon^a [X_a]_B^A = 0. \tag{2.34}$$

As (2.34) must be valid for any $X = \varepsilon^a X_a$, we conclude that it is equivalent to (2.32). ∎

Proposition 2.3. *Let $Q^A(x,t)$ be physical fields satisfying the Euler–Lagrange equations* (2.17)*. If the Lagrangian $\mathcal{L}_0(x,t)$ from* (2.3) *is globally gauge G–invariant then the horizontal and vertical currents satisfy the identities*

$$J_a^{h\alpha}{}_{|\alpha} + J_a^{vi}{}_{\|i} = E_A [X_a]_B^A Q^B, \ \forall\, a \in \{1, ..., r\}. \tag{2.35}$$

Proof. First, multiplying (2.17) by $[X_a]_B^A Q^B$ and taking summation about A, we obtain

$$\left(Q_A^{h\alpha}{}_{|\alpha} + Q_A^{vi}{}_{\|i}\right) [X_a]_B^A Q^B = \left(\frac{\partial \mathcal{L}}{\partial Q^A} - E_A\right) [X_a]_B^A Q^B. \tag{2.36}$$

Then take the horizontal covariant derivative in (2.29) and the vertical covariant derivative in (2.30) and, by adding them, we deduce that

$$\begin{aligned} J_a^{h\alpha}{}_{|\alpha} + J_a^{vi}{}_{\|i} = &- \left(Q_A^{h\alpha}{}_{|\alpha} + Q_A^{vi}{}_{\|i}\right) [X_a]_B^A Q^B \\ &- \left(Q_A^{h\alpha} \frac{\delta Q^B}{\delta x^\alpha} + Q_A^{vi} \frac{\partial Q^B}{\partial t^i}\right) [X_a]_B^A. \end{aligned} \tag{2.37}$$

Finally, by using (2.36) in (2.37) and taking into account (2.32), we obtain (2.35). ∎

The identities (2.35) are called the **conservation laws** for the global gauge invariance of the Lagrangian $\mathcal{L}_0(x,t)$ from (2.3).

As it is well known, many physical theories are developed on a cartesian product of a manifold (sometimes supposed to be compact) and a flat space.

For this reason we think that it is instructive to apply the above theory to a trivial vector bundle ξ whose total space is $E = M \times \mathbb{R}^n$. In this case the coordinate transformations are given by

$$\text{(a) } \widetilde{x}^\alpha = \widetilde{x}^\alpha(x^1, ..., x^p) \quad \text{and} \quad \text{(b) } \widetilde{t}^j = B_i^j t^i, \tag{2.38}$$

where B_i^j are real constants such that $\det[B_i^j] \neq 0$. Suppose that $\mathbb{R}^n$ is equipped with a semi–Euclidean metric $g = [g_{ij}]$ and M carries a semi–Riemannian metric $h = [h_{\alpha\beta}(x)]$. Take on E the semi–Riemannian metric $g \times h$ and from (1.29) obtain $A^i_\alpha = 0$. Thus, from (1.33), we deduce that $C_i{}^k{}_j = 0$, $D_i{}^k{}_\alpha = 0$ and

$$F_\alpha{}^\gamma{}_\beta = \frac{1}{2}\, h^{\gamma\mu}(x) \left\{ \frac{\partial h_{\mu\alpha}}{\partial x^\beta} + \frac{\partial h_{\mu\beta}}{\partial x^\alpha} - \frac{\partial h_{\alpha\beta}}{\partial x^\mu} \right\}. \tag{2.39}$$

Hence, in this case the Vrănceanu connection on E induces the Levi–Civita connections on both M and $\mathbb{R}^n$. Moreover, it is easy to check that the Euler–Lagrange equations (2.17) and the conservation laws (2.35) become

$$\frac{\partial \mathcal{L}}{\partial Q^A} - Q^{h\alpha}_{A}{}_{|\alpha} - Q^{vi}_{A}{}_{\|i} = 0, \tag{2.40}$$

and

$$J^{h\alpha}_{a}{}_{|\alpha} + J^{vi}_{a}{}_{\|i} = 0, \tag{2.41}$$

respectively, since $E_A = 0$ on E for any $A \in \{1, ..., q\}$. In this case, the vertical covariant derivative $Q^{vi}_{A\,\|j}$ reduces to the partial derivative with respect to t^j.

6.3 Local Gauge Invariance on a Vector Bundle

In the present section we suppose that the Lie group G acts locally on the physical fields $Q^A(x,t)$, $A \in \{1, ..., q\}$. This means that the constants ε^a, $a \in \{1, ..., r\}$ from the previous section are now replaced by smooth functions $\varepsilon^a(x,t)$ locally defined on E. Then the **local gauge action** of G on $Q^A(x,t)$ is given by

$$Q'^A(x,t) = Q^A(x,t) + \overset{*}{\delta}(Q^A(x,t)), \tag{3.1}$$

where we put

$$\overset{*}{\delta}(Q^A(x,t)) = \varepsilon^a(x,t)[X_a]^A_B Q^B(x,t). \tag{3.2}$$

In this case we obtain

$$\frac{\delta Q'^A}{\delta x^\alpha} = \frac{\delta Q^A}{\delta x^\alpha} + \varepsilon^a [X_a]^A_B \frac{\delta Q^B}{\delta x^\alpha} + \frac{\delta \varepsilon^a}{\delta x^\alpha} [X_a]^A_B\, Q^B, \tag{3.3}$$

and

$$\frac{\partial Q'^A}{\partial t^i} = \frac{\partial Q^A}{\partial t^i} + \varepsilon^a [X_a]^A_B \frac{\partial Q^B}{\partial t^i} + \frac{\partial \varepsilon^a}{\partial t^i} [X_a]^A_B\, Q^B. \tag{3.4}$$

Thus, if the Lagrangian from (2.3) is globally gauge invariant with respect to (2.23), (2.25) and (2.26), it may fail to be locally gauge invariant with respect to (3.1), (3.3) and (3.4).

In order to obtain a local gauge invariant Lagrangian from a global gauge invariant Lagrangian $\mathcal{L}_0(x,t)$ given by (2.3) we introduce new adapted tensor fields and some special covariant derivatives. First, we suppose that on E there exist r horizontal 1–forms and r vertical 1–forms given locally by

$$\text{(a) } H^a = H^a_\alpha(x,t)dx^\alpha \quad \text{and} \quad \text{(b) } V^a = V^a_i(x,t)\,\delta t^i, \tag{3.5}$$

respectively. We call $\{H^a\}$ and $\{V^a\}$, $a \in \{1,...,r\}$, the **horizontal gauge fields** and the **vertical gauge fields** respectively. Now we assume that the local action of G on the above gauge fields is given by

$$\overset{*}{\delta}(H^a_\alpha(x,t)) = \varepsilon^b(x,t){C_b}^a{}_c H^c_\alpha(x,t) + \frac{\delta\varepsilon^a}{\delta x^\alpha}\,(x,t), \tag{3.6}$$

and

$$\overset{*}{\delta}(V^a_i(x,t)) = \varepsilon^b(x,t){C_b}^a{}_c V^c_i(x,t) + \frac{\partial\varepsilon^a}{\partial t^i}\,(x,t), \tag{3.7}$$

where ${C_b}^a{}_c$ are the structure constants of the Lie algebra of G with respect to the basis $\{X_a\}$, that is we have

$$[X_b, X_c] = {C_b}^a{}_c X_a. \tag{3.8}$$

Elementary properties of the Lie bracket imply that (cf. Helgason, [Hel01], p.136)

$$\begin{aligned} &\text{(a) } {C_b}^a{}_c = -{C_c}^a{}_b, \\ &\text{(b) } {C_b}^e{}_c\,{C_e}^a{}_d + {C_c}^e{}_d\,{C_e}^a{}_b + {C_d}^e{}_b\,{C_e}^a{}_c = 0. \end{aligned} \tag{3.9}$$

On the other hand, for the physical fields $Q^A(x,t)$ we define the **horizontal gauge covariant derivative**

$$D^h_{\frac{\delta}{\delta x^\alpha}} Q^A(x,t) = \frac{\delta Q^A}{\delta x^\alpha}\,(x,t) - H^a_\alpha(x,t)[X_a]^A_B Q^B(x,t), \tag{3.10}$$

and the **vertical gauge covariant derivative**

$$D^v_{\frac{\partial}{\partial t^i}} Q^A(x,t) = \frac{\partial Q^A}{\partial t^i}\,(x,t) - V^a_i(x,t)[X_a]^A_B Q^B(x,t). \tag{3.11}$$

To simplify the notation, we put

$$D^h_\alpha Q^A(x,t) = D^h_{\frac{\delta}{\delta x^\alpha}} Q^A(x,t) \quad \text{and} \quad D^v_i Q^A(x,t) = D^v_{\frac{\partial}{\partial t^i}} Q^A(x,t).$$

Then we prove the following.

Proposition 3.1.

(i) *For each* $A \in \{1, ..., q\}$, $D^h_\alpha Q^A$ *and* $D^v_i Q^A$ *are the local components of the horizontal and vertical* 1*–forms:*

$$\text{(a) } D^h Q^A = D^h_\alpha Q^A dx^\alpha \quad \textit{and} \quad \text{(b) } D^v Q^A = D^v_i Q^A \delta t^i,$$

respectively.

(ii) *The local actions of the Lie group* G *on the gauge covariant derivatives are given by the following homogeneous transformations:*

$$\overset{*}{\delta}(D^h_\alpha Q^A)(x,t) = \varepsilon^a(x,t)[X_a]^A_B D^h_\alpha Q^B(x,t), \tag{3.12}$$

and

$$\overset{*}{\delta}(D^v_i Q^A)(x,t) = \varepsilon^a(x,t)[X_a]^A_B D^v_i Q^B(x,t). \tag{3.13}$$

Proof. From (3.10) we deduce that $D^h_\alpha Q^A$ are the local components of some horizontal 1–forms since $\frac{\delta Q^A}{\delta x^\alpha}$, $A \in \{1, ..., q\}$, and H^a_α, $a \in \{1, ..., r\}$, are so. Similarly, from (3.11) it follows that $D^v_i Q^A$ are the local components of some vertical 1–forms. This proves (i). Next, from (3.3) and (3.4) we infer that

$$\overset{*}{\delta}\left(\frac{\delta Q^A}{\delta x^\alpha}\right) = \varepsilon^a [X_a]^A_B \frac{\delta Q^B}{\delta x^\alpha} + \frac{\delta \varepsilon^a}{\delta x^\alpha}[X_a]^A_B Q^B, \tag{3.14}$$

and

$$\overset{*}{\delta}\left(\frac{\partial Q^A}{\partial t^i}\right) = \varepsilon^a [X_a]^A_B \frac{\partial Q^B}{\partial t^i} + \frac{\partial \varepsilon^a}{\partial t^i}[X_a]^A_B Q^B. \tag{3.15}$$

Then, we apply the local gauge action operator $\overset{*}{\delta}$ to (3.10) and by using (3.14), (3.6) and (3.2), we obtain

$$\begin{aligned} \overset{*}{\delta}(D^h_\alpha Q^A) &= \overset{*}{\delta}\left(\frac{\delta Q^A}{\delta x^\alpha}\right) - \overset{*}{\delta}(H^a_\alpha)[X_a]^A_B Q^B - H^a_\alpha [X_a]^A_B \overset{*}{\delta} Q^B \\ &= \varepsilon^a [X_a]^A_B \frac{\delta Q^B}{\delta x^\alpha} - \varepsilon^b H^c_\alpha C_b{}^a{}_c [X_a]^A_C Q^C - \varepsilon^b H^c_\alpha [X_c]^A_B [X_b]^B_C Q^C. \end{aligned} \tag{3.16}$$

Now, since G has a q–dimensional representation, from (3.8), we deduce that

$$C_b{}^a{}_c [X_a]^A_C = [X_b]^A_B [X_c]^B_C - [X_c]^A_B [X_b]^B_C. \tag{3.17}$$

Thus (3.12) follows from (3.16) by using (3.17). Similarly, by using (3.11), (3.7), (3.2) and (3.15), we obtain (3.13). ■

Next, we consider the Lagrangian

$$\mathcal{L}'_0(x,t) = \mathcal{L}(Q^A(x,t), D^h_\alpha Q^A(x,t), D^v_i Q^A(x,t)), \tag{3.18}$$

where $\mathcal{L}$ is the same function we considered in the global gauge invariant Lagrangian given by (2.3). If $\overset{*}{\delta}\mathcal{L}'_0(x,t) = 0$, then we say that $\mathcal{L}'_0$ is **locally gauge G–invariant**.

Proposition 3.2. *If the Lagrangian $\mathcal{L}_0(x,t)$ from* (2.3) *is globally gauge G–invariant, then $\mathcal{L}'_0(x,t)$ given by* (3.18) *is locally gauge G–invariant.*

Proof. By direct calculations, using (3.2), (3.12) and (3.13), we obtain

$$\begin{aligned}\overset{*}{\delta}\mathcal{L}'_0(x,t) &= \frac{\partial\mathcal{L}}{\partial Q^A}\,\overset{*}{\delta}Q^A + \frac{\partial\mathcal{L}}{\partial(D^h_\alpha Q^A)}\,\overset{*}{\delta}(D^h_\alpha Q^A) + \frac{\partial\mathcal{L}}{\partial(D^v_i Q^A)}\,\overset{*}{\delta}(D^v_i Q^A)\\ &= \left(\frac{\partial\mathcal{L}}{\partial Q^A}\,Q^B + \frac{\partial\mathcal{L}}{\partial(D^h_\alpha Q^A)}\,D^h_\alpha Q^B + \frac{\partial\mathcal{L}}{\partial(D^v_i Q^A)}\,D^v_i Q^B\right)[X_a]^A_B\varepsilon^a.\end{aligned}$$

Then, taking into account (2.32), we deduce that $\overset{*}{\delta}\mathcal{L}'_0(x,t) = 0$, that is, $\mathcal{L}'_0(x,t)$ is locally gauge G–invariant. ∎

In conclusion, we may say that from a globally gauge G–invariant Lagrangian $\mathcal{L}_0(x,t)$ we obtain a locally gauge G–invariant Lagrangian by a simple replacement of $\dfrac{\delta Q^A}{\delta x^\alpha}$ and $\dfrac{\partial Q^A}{\partial t^i}$ from $\mathcal{L}_0$ by $D^h_\alpha Q^A$ and $D^v_i Q^A$ respectively.

Now, by means of the gauge fields and the Vrănceanu connection on E, we define locally the following functions

$$R^a{}_{\alpha\beta} = \frac{\delta H^a_\alpha}{\delta x^\beta} - \frac{\delta H^a_\beta}{\delta x^\alpha} - C_b{}^a{}_c H^c_\alpha H^b_\beta + T_\alpha{}^i{}_\beta V^a_i, \tag{3.19}$$

$$P^a{}_{\alpha i} = \frac{\partial H^a_\alpha}{\partial t^i} - \frac{\delta V^a_i}{\delta x^\alpha} - C_b{}^a{}_c H^c_\alpha V^b_i + D_i{}^k{}_\alpha V^a_k, \tag{3.20}$$

$$S^a{}_{ij} = \frac{\partial V^a_i}{\partial t^j} - \frac{\partial V^a_j}{\partial t^i} - C_b{}^a{}_c V^c_i V^b_j, \tag{3.21}$$

where $T_\alpha{}^i{}_\beta$ and $D_i{}^k{}_\alpha$ are given by (1.15) and (1.33b) respectively.

Proposition 3.3. *For each $a \in \{1,...,r\}$, the functions $R^a{}_{\alpha\beta}$, $P^a{}_{\alpha i}$ and $S^a{}_{ij}$ define the adapted tensor fields R^a, P^a and S^a of type* $(0,0;0,2)$, $(0,1;0,1)$ *and* $(0,2;0,0)$ *on E, respectively.*

Proof. By using (1.13a) for the horizontal gauge fields, we obtain

$$\frac{\delta H^a_\alpha}{\delta x^\beta} = \frac{\delta \widetilde{H}^a_\gamma}{\delta \widetilde{x}^\varepsilon}\,J^\gamma_\alpha J^\varepsilon_\beta + \widetilde{H}^a_\gamma\,\frac{\partial^2\widetilde{x}^\gamma}{\partial x^\alpha\partial x^\beta}, \tag{3.22}$$

with respect to (1.1). Next, we apply (1.13b) for the vertical gauge fields V^a_i and (1.14) for $T_\alpha{}^i{}_\beta$, and obtain

$$T_{\alpha}{}^{i}{}_{\beta}V_i^a = T_{\alpha}{}^{i}{}_{\beta}B_i^j\widetilde{V}_j^a = \widetilde{T}_{\gamma}{}^{j}{}_{\varepsilon}J_{\alpha}^{\gamma}J_{\beta}^{\varepsilon}\widetilde{V}_j^a. \tag{3.23}$$

Then, by using (3.22) and (3.23) in (3.19), we deduce that

$$R^a{}_{\alpha\beta} = \widetilde{R}^a{}_{\gamma\varepsilon}J_{\alpha}^{\gamma}J_{\beta}^{\varepsilon},$$

that is, $R^a{}_{\alpha\beta}$ define an adapted tensor field of type $(0,0;0,2)$. Now, applying the operators $\dfrac{\partial}{\partial t^i}$ and $\dfrac{\delta}{\delta x^{\alpha}}$ to (1.13a) and (1.13b) for H_{α}^a and V_i^a respectively, and by using (1.7) and (1.2a), we infer that

$$\frac{\partial H_{\alpha}^a}{\partial t^i} = \frac{\partial \widetilde{H}_{\beta}^a}{\partial \widetilde{t}^j}\, B_i^j J_{\alpha}^{\beta}, \tag{3.24}$$

and

$$\frac{\delta V_i^a}{\delta x^{\alpha}} = \frac{\delta \widetilde{V}_j^a}{\delta \widetilde{x}^{\gamma}}\, J_{\alpha}^{\gamma}B_i^j + \widetilde{V}_j^a\, \frac{\delta}{\delta x^{\alpha}}\,(B_i^j). \tag{3.25}$$

We follow the transformations (2.3.11) for $D_i{}^k{}_{\alpha}$ and obtain

$$D_i{}^k{}_{\alpha}V_k^a = D_i{}^k{}_{\alpha}B_k^j\widetilde{V}_j^a = \widetilde{D}_h{}^j{}_{\beta}B_i^h J_{\alpha}^{\beta}\widetilde{V}_j^a + \widetilde{V}_j^a\, \frac{\delta}{\delta x^{\alpha}}\,(B_i^j). \tag{3.26}$$

By direct calculations, using (3.24)–(3.26) into (3.20), we deduce that $P^a{}_{\alpha i}$ are the components of an adapted tensor field of type $(0,1;0,1)$. Similarly, it follows that $S^a{}_{ij}$ define an adapted tensor field of type $(0,2;0,0)$ for any $a \in \{1,...,r\}$. ∎

We call the tensor fields $R^a = (R^a{}_{\alpha\beta})$, $P^a = (P^a{}_{\alpha i})$ and $S^a = (S^a{}_{ij})$, $a \in \{1,...,r\}$, the **horizontal**, **mixed** and **vertical strength fields** for the gauge theory we develop on the total space E of the vector bundle ξ.

Now, in order to construct some Lagrangians for the gauge fields $H^a = (H_{\alpha}^a)$ and $V^a = (V_i^a)$ we suppose that E is endowed with a semi–Riemannian metric g, and VE is a semi–Riemannian distribution with respect to g. As in Section 6.1, we denote by $\{g_{ij}\}$ and $\{h_{\alpha\beta}\}$ (see (1.30) and (1.31)) the local components of the semi–Riemannian metrics induced by g on VE and HE respectively. Also, we need some concepts and results from the theory of Lie algebras. First, for any $X \in G^*$, we have the linear transformation

$$\operatorname{ad} X : G^* \to G^*,\ (\operatorname{ad} X)(Y) = [X,Y],\quad \forall\, Y \in G^*. \tag{3.27}$$

It is easy to check that $\operatorname{ad} X$ is a homomorphism of the Lie algebra G^*. Hence $X \to \operatorname{ad} X$ is a representation of G^* on G^*. In the literature this representation is known as the **adjoint representation** of G^*. Then, we define the mapping

$$K : G^*{\times}G^* \to \mathbb{R};\ K(X,Y) = \operatorname{Tr}(\operatorname{ad} X\ \operatorname{ad} Y),\quad \forall\, X,Y \in G^*, \tag{3.28}$$

where Tr represents the trace operator. It is easy to see that K is a symmetric bilinear form on G^*. Moreover, K satisfies (cf. Helgason [Hel01], p.131)

$$K(X,[Y,Z]) = K(Y,[Z,X]) = K(Z,[X,Y]), \tag{3.29}$$

for any $X, Y, Z \in G^*$. The form K is called the **Killing form** of G^*. If $C_b{}^a{}_c$ are the structure constants of G^* with respect to the basis $\{X_a\}$, $a \in \{1, ..., r\}$, then from (3.27) and (3.28) it follows that K is given by the matrix $[K_{ab}]$, where

$$K_{ab} = C_a{}^c{}_d \, C_b{}^d{}_c. \tag{3.30}$$

Also, (3.29) is equivalent to

$$K_{ad} \, C_b{}^d{}_c = K_{bd} \, C_c{}^d{}_a = K_{cd} \, C_a{}^d{}_b. \tag{3.31}$$

If K is non–degenerate, then G^* (resp. G) is called a **semisimple Lie algebra** (resp. **Lie group**). From now on we consider that G is a semisimple compact Lie group. In this case, the Killing form is negative definite.

Next, on each coordinate neighbourhood of E we define the smooth functions:

$$L_H(x,t) = -\frac{1}{4} \, K_{ab} h^{\alpha\beta}(x,t) h^{\gamma\varepsilon}(x,t) R^a{}_{\alpha\gamma}(x,t) R^b{}_{\beta\varepsilon}(x,t), \tag{3.32}$$

$$L_{HV}(x,t) = -\frac{1}{2} \, K_{ab} h^{\alpha\beta}(x,t) g^{ij}(x,t) P^a{}_{\alpha i}(x,t) P^b{}_{\beta j}(x,t), \tag{3.33}$$

$$L_V(x,t) = -\frac{1}{4} \, K_{ab} g^{ij}(x,t) g^{hk}(x,t) S^a{}_{ih}(x,t) S^b{}_{jk}(x,t). \tag{3.34}$$

Taking into account that $h^{\alpha\beta}$ and g^{ij} are the local components of some adapted tensor fields of type $(0,0;2,0)$ and $(2,0;0,0)$ respectively, and using Proposition 3.3, we conclude that L_H, L_{HV} and L_V define three Lagrangians on E which we call the **horizontal**, **mixed** and **vertical Lagrangian** respectively, for the gauge fields H^a and V^a. Moreover, we prove the following important result.

Theorem 3.4. *The horizontal, mixed and vertical Lagrangians for the gauge fields on the total space of a vector bundle are locally gauge G–invariant.*

Proof. First, by using (3.6), we obtain

$$\overset{*}{\delta}\left(\frac{\delta H^a_\alpha}{\delta x^\beta}\right) = \frac{\delta \varepsilon^b}{\delta x^\beta} \, C_b{}^a{}_c H^c_\alpha + \varepsilon^b \, C_b{}^a{}_c \, \frac{\delta H^c_\alpha}{\delta x^\beta} + \frac{\delta}{\delta x^\beta} \left(\frac{\delta \varepsilon^a}{\delta x^\alpha}\right), \tag{3.35}$$

and

$$\begin{aligned} &\overset{*}{\delta}(C_b{}^a{}_c H^c_\alpha H^b_\beta) \\ &= \varepsilon^b (C_e{}^c{}_b C_c{}^a{}_d + C_b{}^c{}_d C_c{}^a{}_e) H^e_\alpha H^d_\beta + C_b{}^a{}_c \left(\frac{\delta \varepsilon^c}{\delta x^\alpha} \, H^b_\beta + \frac{\delta \varepsilon^b}{\delta x^\beta} \, H^c_\alpha\right). \end{aligned} \tag{3.36}$$

By the identity (3.9b), (3.36) becomes

$$\overset{*}{\delta}(C_b{}^a{}_c H^c_\alpha H^b_\beta) = \varepsilon^b C_b{}^a{}_c C_d{}^c{}_e H^e_\alpha H^d_\beta + C_b{}^a{}_c \left(\frac{\delta \varepsilon^c}{\delta x^\alpha} H^b_\beta + \frac{\delta \varepsilon^b}{\delta x^\beta} H^c_\alpha \right) \cdot \quad (3.37)$$

On the other hand, by (3.7) we deduce that

$$T_\alpha{}^i{}_\beta \overset{*}{\delta} V^a_i = \varepsilon^b C_b{}^a{}_c T_\alpha{}^i{}_\beta V^c_i + T_\alpha{}^i{}_\beta \frac{\partial \varepsilon^a}{\partial t^i} \cdot \quad (3.38)$$

Now, applying the local gauge operator $\overset{*}{\delta}$ to (3.19) and by using (3.35), (3.37), (3.38) and (1.16) we obtain

$$\overset{*}{\delta} R^a{}_{\alpha\beta} = \varepsilon^b C_b{}^a{}_c R^c{}_{\alpha\beta}. \quad (3.39)$$

Similar calculations for the mixed and vertical strength fields lead us to the following transformations

$$\text{(a) } \overset{*}{\delta} P^a{}_{\alpha i} = \varepsilon^b C_b{}^a{}_c P^c{}_{\alpha i}, \quad \text{(b) } \overset{*}{\delta} S^a{}_{ij} = \varepsilon^b C_b{}^a{}_c S^c{}_{ij}. \quad (3.40)$$

Now, we apply $\overset{*}{\delta}$ to L_H and taking into account (3.39) we infer that

$$\overset{*}{\delta} L_H = -\frac{1}{4} h^{\alpha\beta} h^{\gamma\mu} \varepsilon^d (K_{ab} C_d{}^b{}_e + K_{eb} C_d{}^b{}_a) R^a{}_{\alpha\gamma} R^e{}_{\beta\mu}. \quad (3.41)$$

Then, by using (3.31) and (3.9a) in (3.41), we deduce that $\overset{*}{\delta} L_H = 0$, which means that L_H is locally gauge G–invariant. In a similar way, by using (3.40, (3.31) and (3.9a), we obtain $\overset{*}{\delta} L_{HV} = \overset{*}{\delta} L_V = 0$. ∎

6.4 Equations of Motion and Conservation Laws

In the present section we consider the Lagrangian

$$\mathcal{L}(x,t) = \mathcal{L}'_0(x,t) + L_H(x,t) + L_{HV}(x,t) + L_V(x,t), \quad (4.1)$$

where $\mathcal{L}'_0$ is given by (3.18) and L_H, L_{HV} and L_V are the Lagrangians for gauge fields given by (3.32), (3.33) and (3.34) respectively. By Proposition 3.2 and Theorem 3.4 we deduce that $\mathcal{L}(x,t)$ given by (4.1) is locally gauge G–invariant. Thus $\mathcal{L}(x,t)$ can be proposed as **full Lagrangian** for the gauge theory we want to develop on the total space E of the vector bundle $\xi = (E, \pi, M)$. To this end we define the Lagrangian density

$$\mathcal{L}^*(x,t) = \mathcal{L}(x,t) H(x,t) V(x,t), \quad (4.2)$$

where $H(x,t)$ and $V(x,t)$ are given by (2.6). Then we consider the variational principle

$$\delta\left(\int_{\Omega}\mathcal{L}^*(x,t)dx^1\wedge\cdots\wedge dx^p\wedge dt^1\wedge\cdots\wedge dt^n\right)=0. \tag{4.3}$$

The same principle was considered by Asanov [Asa85], p.244, but with respect to some other Lagrangians. As $\mathcal{L}^*(x,t)$ contains the physical fields $Q^A(x,t)$ and the gauge fields $H^a_\alpha(x,t)$ and $V^a_i(x,t)$, we have the following Euler–Lagrange equations:

$$\frac{\partial\mathcal{L}^*}{\partial Q^A}-\frac{\partial}{\partial x^\alpha}\left(\frac{\partial\mathcal{L}^*}{\partial\left(\dfrac{\partial Q^A}{\partial x^\alpha}\right)}\right)-\frac{\partial}{\partial t^i}\left(\frac{\partial\mathcal{L}^*}{\partial\left(\dfrac{\partial Q^A}{\partial t^i}\right)}\right)=0, \tag{4.4}$$

$$\frac{\partial\mathcal{L}^*}{\partial H^a_\alpha}-\frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}^*}{\partial\left(\dfrac{\partial H^a_\alpha}{\partial x^\beta}\right)}\right)-\frac{\partial}{\partial t^i}\left(\frac{\partial\mathcal{L}^*}{\partial\left(\dfrac{\partial H^a_\alpha}{\partial t^i}\right)}\right)=0, \tag{4.5}$$

$$\frac{\partial\mathcal{L}^*}{\partial V^a_i}-\frac{\partial}{\partial x^\alpha}\left(\frac{\partial\mathcal{L}^*}{\partial\left(\dfrac{\partial V^a_i}{\partial x^\alpha}\right)}\right)-\frac{\partial}{\partial t^j}\left(\frac{\partial\mathcal{L}^*}{\partial\left(\dfrac{\partial V^a_i}{\partial t^j}\right)}\right)=0. \tag{4.6}$$

According to the theory we developed in Section 6.2, the equations in (4.4) can be expressed as follows (see (2.17))

$$\frac{\partial\mathcal{L}}{\partial Q^A}-Q^{h\alpha}_{A}{}_{|\alpha}-Q^{vi}_{A}{}_{\|i}=E_A, \tag{4.7}$$

where $Q^{h\alpha}_A$, Q^{vi}_A and E_A are given by (2.13), (2.14) and (2.18) respectively. It is also important to mention that the covariant derivatives in (4.7) are taken with respect to the Vrănceanu connection given by (1.33). We look now for similar expressions as in (4.7) but for (4.5) and (4.6). To this end we first put

$$\text{(a)}\ H^{h\alpha\beta}_a=\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\delta H^a_\alpha}{\delta x^\beta}\right)},\quad \text{(b)}\ H^{v\alpha i}_a=\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial H^a_\alpha}{\partial t^i}\right)}\star, \tag{4.8}$$

and

$$\text{(a)}\ V^{hi\alpha}_a=\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\delta V^a_i}{\delta x^\alpha}\right)},\quad \text{(b)}\ V^{vij}_a=\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial V^a_i}{\partial t^j}\right)}\star, \tag{4.9}$$

where $\star$ in (4.8b) and (4.9b) means that we take partial derivatives of $\mathcal{L}$ only with respect to the variables $\dfrac{\partial H^a_\alpha}{\partial t^i}$ and $\dfrac{\partial V^a_i}{\partial t^j}$ which are not incorporated in the expressions of $\dfrac{\delta H^a_\alpha}{\delta x^\beta}$ and $\dfrac{\delta V^a_i}{\delta x^\alpha}$ respectively.

Proposition 4.1. *The smooth functions* $H_a^{h\alpha\beta}$, $H_a^{v\alpha i}$, $V_a^{hi\alpha}$, V_a^{vij} *define adapted tensor fields on* E *of type* $(0,0;2,0)$, $(1,0;1,0)$, $(1,0;1,0)$ *and* $(2,0;0,0)$ *respectively, for any* $a \in \{1,...,r\}$.

Proof. By direct calculations using (3.22), we obtain

$$\widetilde{H}_a^{h\gamma\varepsilon} = \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\delta \widetilde{H}_\gamma^a}{\delta \widetilde{x}^\varepsilon}\right)} = \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\delta H_\alpha^a}{\delta x^\beta}\right)} \frac{\partial\left(\dfrac{\delta H_\alpha^a}{\delta x^\beta}\right)}{\partial\left(\dfrac{\delta \widetilde{H}_\gamma^a}{\delta \widetilde{x}^\varepsilon}\right)} = H_a^{h\alpha\beta} J_\alpha^\gamma J_\beta^\varepsilon,$$

with respect to the coordinate transformations (1.1) on E. Hence, $H_a^{h\alpha\beta}$ define an adapted tensor field of type $(0,0;2,0)$ on E for each $a \in \{1,...,r\}$. Next, by (3.24) we deduce that

$$\widetilde{H}_a^{v\gamma j} = \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial \widetilde{H}_\gamma^a}{\partial \widetilde{t}^j}\right)} \star = \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial H_\alpha^a}{\partial t^i}\right)} \star \frac{\partial\left(\dfrac{\partial H_\alpha^a}{\partial t^i}\right)}{\partial\left(\dfrac{\partial \widetilde{H}_\gamma^a}{\partial t^j}\right)} = H_a^{v\alpha i} B_i^j J_\alpha^\gamma.$$

Thus, for each $a \in \{1,...,r\}$, $H_a^{v\alpha i}$ define an adapted tensor field of type $(1,0;1,0)$ on E. By similar calculations it follows that $V_a^{hi\alpha}$ and V_a^{vij} define adapted tensor fields of type $(1,0;1,0)$ and $(2,0;0,0)$ respectively. ∎

Moreover, $H_a^{h\alpha\beta}$ and V_a^{vij} define skew–symmetric adapted tensor fields on E. Indeed, by using (3.19) and (3.21) we deduce that

$$\text{(a) } H_a^{h\alpha\beta} = 2\,\frac{\partial \mathcal{L}}{\partial R^a{}_{\alpha\beta}} \quad \text{and} \quad \text{(b) } V_a^{vij} = 2\,\frac{\partial \mathcal{L}}{\partial S^a{}_{ij}}\cdot \tag{4.10}$$

Then $H_a^{h\alpha\beta}$ and V_a^{vij} are skew–symmetric since $R^a{}_{\alpha\beta}$ and $S^a{}_{ij}$ are so.

Proposition 4.2. *The Euler–Lagrange equations* (4.5) *and* (4.6) *can be written as follows*

$$\frac{\partial \mathcal{L}}{\partial H_\alpha^a} - H_a^{h\alpha\beta}{}_{|\beta} - H_a^{v\alpha i}{}_{\|i} = E_a^{h\alpha}, \tag{4.11}$$

and

$$\frac{\partial \mathcal{L}}{\partial V_i^a} + D_j{}^i{}_\alpha V_a^{hj\alpha} - V_a^{hi\alpha}{}_{|\alpha} - V_a^{vij}{}_{\|j} = E_a^{vi}, \tag{4.12}$$

where the horizontal and vertical covariant derivatives are taken with respect to the Vrănceanu connection, and we put

$$\begin{aligned} E_a^{h\alpha} = {} & \left\{ \frac{1}{HV}\,\frac{\delta(HV)}{\delta x^\beta} - \left(D_i{}^i{}_\beta + F_\beta{}^\gamma{}_\gamma\right) \right\} H_a^{h\alpha\beta} \\ & + \left\{ \frac{1}{HV}\,\frac{\partial(HV)}{\partial t^i} - C_i{}^j{}_j \right\} H_a^{v\alpha i}, \end{aligned} \tag{4.13}$$

and

$$E_a^{vi} = \left\{ \frac{1}{HV} \frac{\delta(HV)}{\delta x^\alpha} - \left(D_j{}^j{}_\alpha + F_\alpha{}^\gamma{}_\gamma \right) \right\} V_a^{hi\alpha} + \left\{ \frac{1}{HV} \frac{\partial(HV)}{\partial t^k} - C_k{}^j{}_j \right\} V_a^{vik}. \tag{4.14}$$

Proof. First, by using (4.2), (1.5) and (4.8), we obtain

$$\frac{\partial \mathcal{L}^*}{\partial \left(\dfrac{\partial H_\alpha^a}{\partial x^\beta} \right)} = H_a^{h\alpha\beta} HV, \tag{4.15}$$

and

$$\frac{\partial \mathcal{L}^*}{\partial \left(\dfrac{\partial H_\alpha^a}{\partial t^i} \right)} = \left\{ H_a^{v\alpha i} - A_\beta^i H_a^{h\alpha\beta} \right\} HV. \tag{4.16}$$

Next, the horizontal covariant derivative of $H_a^{h\alpha\beta}$ with respect to the Vrănceanu connection is given by (see Section 6.1)

$$H_a^{h\alpha\beta}{}_{|\gamma} = \frac{\delta H_a^{h\alpha\beta}}{\delta x^\gamma} + H_a^{h\varepsilon\beta} F_\varepsilon{}^\alpha{}_\gamma + H_a^{h\alpha\varepsilon} F_\varepsilon{}^\beta{}_\gamma.$$

By contraction over β and γ, and taking into account that $H_a^{h\varepsilon\beta}$ are skew–symmetric while $F_\varepsilon{}^\alpha{}_\gamma$ are symmetric (see 1.33d), we deduce that

$$H_a^{h\alpha\beta}{}_{|\beta} = \frac{\delta H_a^{h\alpha\beta}}{\delta x^\beta} + H_a^{h\alpha\varepsilon} F_\varepsilon{}^\beta{}_\beta. \tag{4.17}$$

Also, by using (1.33c), we obtain

$$H_a^{v\alpha i}{}_{\|i} = \frac{\partial H_a^{v\alpha i}}{\partial t^i} + H_a^{v\alpha k} C_k{}^i{}_i. \tag{4.18}$$

Now replace the partial derivatives of $\mathcal{L}^*$ from (4.15) and (4.16) into (4.5) and by a lengthy (but not difficult) calculation using (4.17), (4.18) and (1.33b) we derive (4.11). Similar calculations lead us to 4.12. ■

As a consequence of the above theorem we may see that (4.7), (4.11) and (4.12) are the **equations of motion** with respect to the variational principle (4.3) on the total space E of the vector bundle ξ.

To obtain the corresponding conservation laws we first note that the full Lagrangian $\mathcal{L}(x,t)$ given by (4.1) is locally gauge G–invariant, that is, we have $\overset{*}{\delta}\mathcal{L}(x,t) = 0$. Then, by using (2.13), (2.14), (4.8) and (4.9), we obtain

$$
\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial Q^A}\overset{*}{\delta}(Q^A) + Q_A^{h\alpha}\overset{*}{\delta}\left(\frac{\delta Q^A}{\delta x^\alpha}\right) + Q_A^{vi}\overset{*}{\delta}\left(\frac{\partial Q^A}{\partial t^i}\right)\\
&+\frac{\partial \mathcal{L}}{\partial H_\alpha^a}\overset{*}{\delta}(H_\alpha^a) + H_a^{h\alpha\beta}\overset{*}{\delta}\left(\frac{\delta H_\alpha^a}{\delta x^\beta}\right) + H_a^{v\alpha i}\overset{*}{\delta}\left(\frac{\partial H_\alpha^a}{\partial t^i}\right)\\
&+\frac{\partial \mathcal{L}}{\partial V_i^a}\overset{*}{\delta}(V_i^a) + V_a^{hi\alpha}\overset{*}{\delta}\left(\frac{\delta V_i^a}{\delta x^\alpha}\right) + V_a^{vij}\overset{*}{\delta}\left(\frac{\partial V_i^a}{\partial t^j}\right) = 0.
\end{aligned}
\tag{4.19}
$$

Taking into account that $\overset{*}{\delta}$ commutes with both $\dfrac{\delta}{\delta x^\alpha}$ and $\dfrac{\partial}{\partial t^i}$, and by using the equations of motion, we deduce that

$$
\begin{aligned}
&\frac{\delta}{\delta x^\beta}\left(Q_A^{h\beta}\overset{*}{\delta}(Q^A) + H_a^{h\alpha\beta}\overset{*}{\delta}(H_\alpha^a) + V_a^{hi\beta}\overset{*}{\delta}(V_i^a)\right)\\
&+\frac{\partial}{\partial t^j}\left(Q_A^{vj}\overset{*}{\delta}(Q^A) + H_a^{v\alpha j}\overset{*}{\delta}(H_\alpha^a) + V_a^{vij}\overset{*}{\delta}(V_i^a)\right)\\
&+\left(Q_A^{h\beta}\overset{*}{\delta}(Q^A) + H_a^{h\alpha\beta}\overset{*}{\delta}(H_\alpha^a) + V_a^{hi\beta}\overset{*}{\delta}(V_i^a)\right)F_\beta{}^\gamma{}_\gamma\\
&+\left(Q_A^{vj}\overset{*}{\delta}(Q^A) + H_a^{v\alpha j}\overset{*}{\delta}(H_\alpha^a) + V_a^{vij}\overset{*}{\delta}(V_i^a)\right)C_j{}^k{}_k\\
&+E_A\overset{*}{\delta}(Q^A) + E_a^{h\alpha}\overset{*}{\delta}(H_\alpha^a) + E_a^{vi}\overset{*}{\delta}(V_i^a) = 0.
\end{aligned}
\tag{4.20}
$$

Next, we replace $\overset{*}{\delta}(Q^A), \overset{*}{\delta}(H_\alpha^a)$ and $\overset{*}{\delta}(V_i^a)$ from (3.2), (3.6) and (3.7) respectively to (4.20) and arrange it as follows

$$
\begin{aligned}
&\Big\{E_A[X_a]_B^A Q^B + E_b^{h\alpha}C_a{}^b{}_c H_\alpha^c + E_b^{vi}C_a{}^b{}_c V_i^c\\
&\quad -\frac{\delta}{\delta x^\beta}(J_a^{h\beta}) - \frac{\partial}{\partial t^j}(J_a^{vj}) - J_a^{h\beta}F_\beta{}^\gamma{}_\gamma - J_a^{vj}C_j{}^k{}_k\Big\}\varepsilon^a\\
&+\left\{\frac{\delta H_a^{h\beta\alpha}}{\delta x^\alpha} + \frac{\partial H_a^{v\beta j}}{\partial t^j} - J_a^{h\beta} + H_a^{h\beta\gamma}F_\gamma{}^\alpha{}_\alpha + H_a^{v\beta k}C_k{}^j{}_j + E_a^{h\beta}\right\}\frac{\delta\varepsilon^a}{\delta x^\beta}\\
&+\left\{\frac{\delta V_a^{hj\beta}}{\delta x^\beta} + \frac{\partial V_a^{vji}}{\partial t^i} - J_a^{vj} + V_a^{hj\gamma}F_\gamma{}^\beta{}_\beta + V_a^{vjk}C_k{}^i{}_i + E_a^{vj}\right\}\frac{\partial\varepsilon^a}{\partial t^j}\\
&+\Big\{H_a^{h\alpha\beta}\frac{\delta}{\delta x^\beta}\left(\frac{\delta\varepsilon^a}{\delta x^\alpha}\right) + V_a^{hi\alpha}\frac{\delta}{\delta x^\alpha}\left(\frac{\partial\varepsilon^a}{\partial t^i}\right)\\
&\quad + H_a^{v\alpha i}\frac{\partial}{\partial t^i}\left(\frac{\delta\varepsilon^a}{\delta x^\alpha}\right) + V_a^{vij}\frac{\partial^2\varepsilon^a}{\partial t^j\partial t^i}\Big\} = 0,
\end{aligned}
\tag{4.21}
$$

where we put

$$
J_a^{h\beta} = -Q_A^{h\beta}[X_a]_B^A Q^B - H_b^{h\alpha\beta}C_a{}^b{}_c H_\alpha^c - V_b^{hi\beta}C_a{}^b{}_c V_i^c, \tag{4.22}
$$

and

$$J_a^{vj} = -Q_A^{vj}[X_a]_B^A Q^B - H_b^{v\alpha j} C_a{}^b{}_c H_\alpha^c - V_b^{vij} C_a{}^b{}_c V_i^c. \tag{4.23}$$

Now, we examine the terms between the last brackets { } in (4.21). First, taking into account that $H_a^{h\alpha\beta}$ is a skew–symmetric adapted tensor field for any $a \in \{1, ..., r\}$, and by using (1.16), we obtain

$$\begin{aligned} H_a^{h\alpha\beta} \frac{\delta}{\delta x^\beta}\left(\frac{\delta\varepsilon^a}{\delta x^\alpha}\right) &= \frac{1}{2} H_a^{h\alpha\beta}\left(\frac{\delta}{\delta x^\beta}\left(\frac{\delta\varepsilon^a}{\delta x^\alpha}\right) - \frac{\delta}{\delta x^\alpha}\left(\frac{\delta\varepsilon^a}{\delta x^\beta}\right)\right) \\ &= \frac{1}{2} H_a^{h\alpha\beta} T_\beta{}^j{}_\alpha \frac{\partial\varepsilon^a}{\partial t^j}\,. \end{aligned} \tag{4.24}$$

Then, by using (1.35), we deduce that

$$\begin{aligned} & V_a^{hi\alpha} \frac{\delta}{\delta x^\alpha}\left(\frac{\partial\varepsilon^a}{\partial t^i}\right) + H_a^{v\alpha i} \frac{\partial}{\partial t^i}\left(\frac{\delta\varepsilon^a}{\delta x^\alpha}\right) \\ & \quad = \left(V_a^{hi\alpha} + H_a^{v\alpha i}\right) \frac{\partial}{\partial t^i}\left(\frac{\delta\varepsilon^a}{\delta x^\alpha}\right) + V_a^{hi\alpha} D_i{}^j{}_\alpha \frac{\partial\varepsilon^a}{\partial t^j}\,. \end{aligned} \tag{4.25}$$

Finally, since V_a^{vij} is a skew–symmetric adapted tensor field for any $a \in \{1,...,r\}$, we have

$$V_a^{vij} \frac{\partial^2\varepsilon^a}{\partial t^j \partial t^i} = 0. \tag{4.26}$$

By using (4.24)–(4.26) into (4.21) and taking into account the arbitrariness of ε^a, we obtain the following:

$$\begin{aligned} & E_A[X_a]_B^A Q^B + E_b^{h\alpha} C_a{}^b{}_c H_\alpha^c + E_b^{vi} C_a{}^b{}_c V_i^c \\ & \quad - \frac{\delta}{\delta x^\beta}\left(J_a^{h\beta}\right) - \frac{\partial}{\partial t^j}\left(J_a^{vj}\right) - J_a^{h\gamma} F_\gamma{}^\beta{}_\beta - J_a^{vi} C_i{}^j{}_j = 0, \end{aligned} \tag{4.27}$$

$$\frac{\delta H_a^{h\beta\alpha}}{\delta x^\alpha} + \frac{\partial H_a^{v\beta j}}{\partial t^j} - J_a^{h\beta} + H_a^{h\beta\gamma} F_\gamma{}^\alpha{}_\alpha + H_a^{v\beta k} C_k{}^j{}_j + E_a^{h\beta} = 0, \tag{4.28}$$

$$\begin{aligned} & \frac{\delta V_a^{hj\beta}}{\delta x^\beta} + \frac{\partial V_a^{vji}}{\partial t^i} - J_a^{vj} + V_a^{hj\gamma} F_\gamma{}^\alpha{}_\alpha \\ & \quad + V_a^{vjk} C_k{}^i{}_i + E_a^{vj} + \frac{1}{2} H_a^{h\alpha\beta} T_\beta{}^j{}_\alpha + V_a^{hi\alpha} D_i{}^j{}_\alpha = 0, \end{aligned} \tag{4.29}$$

and

$$V_a^{hi\alpha} + H_a^{v\alpha i} = 0. \tag{4.30}$$

Then, by using (4.11), (4.17) and (4.18) in (4.28), we infer that

$$J_a^{h\beta} = \frac{\partial\mathcal{L}}{\partial H_\beta^a}\,. \tag{4.31}$$

In a similar way, from (4.29) we obtain

$$J_a^{vj} = \frac{\partial \mathcal{L}}{\partial V_j^a} + \frac{1}{2} H_a^{h\alpha\beta} T_\beta{}^i{}_\alpha + V_a^{hi\alpha} D_i{}^j{}_\alpha. \tag{4.32}$$

Thus the new formulas (4.31) and (4.32) for $J_a^{h\beta}$ and J_a^{vj} do not contain the structure constants of the Lie group G which are present in (4.22) and (4.23). On the other hand, taking into account Proposition 4.1, from (4.22) and (4.23) we deduce that $J_a^{h\beta}$ and J_a^{vj} define a horizontal vector field and a vertical vector field for each $a \in \{1, ..., r\}$. We call

$$\text{(a) } J_a^h = J_a^{h\beta} \frac{\delta}{\delta x^\beta} \quad \text{and} \quad \text{(b) } J_a^v = J_a^{vj} \frac{\partial}{\partial t^j}, \tag{4.33}$$

the **horizontal currents** and **vertical currents** respectively, corresponding to the full Lagrangian $\mathcal{L}(x, t)$. Finally, we state the following.

Theorem 4.3. *The conservation laws for the local gauge action of the Lie group G with respect to the variational principle* (4.3) *are given by*

$$J_a^{h\beta}{}_{|\beta} + J_a^{vj}{}_{\|j} = E_A [X_a]_B^A Q^B + E_b^{h\alpha} C_a{}^b{}_c H_\alpha^c + E_a^{vi} C_a{}^b{}_c V_i^c, \tag{4.34}$$

where the horizontal and vertical covariant derivatives are taken with respect to the Vrănceanu connection.

Proof. It follows from (4.27) by using (1.20) and (1.23). ■

In concluding this section we apply the above gauge theory to a full Lagrangian $\mathcal{L}(x, y)$ on a trivial vector bundle ξ with total space $M \times \mathbb{R}^n$, where M is a p–dimensional manifold. As we have seen in Section 6.2, $E_A = 0$ for any $A \in \{1, ..., q\}$. Also we have $C_i{}^k{}_j = 0$ and $D_i{}^k{}_\alpha = 0$. Finally, by using (2.39) and (2.6a), we deduce that

$$\frac{1}{H} \frac{\delta H}{\delta x^\beta} = \frac{1}{H} \frac{\partial H}{\partial x^\beta} = F_\beta{}^\gamma{}_\gamma.$$

Thus, in this particular case, $E_a^{h\alpha}$ and E_a^{vi} from (4.13) and (4.14) vanish on $M \times \mathbb{R}^n$ for all $a \in \{1, ..., r\}$, $\alpha \in \{1, ..., p\}$ and $i \in \{1, ..., n\}$. Then, by Proposition 4.2 and Theorem 4.3, we may state the following.

Theorem 4.4. *Let $\mathcal{L}(x, t)$ be a full Lagrangian given by* (4.1) *on $M \times \mathbb{R}^n$. Then we have the following assertions:*

(i) *The equations of motion are given by*

$$\frac{\partial \mathcal{L}}{\partial Q^A} - Q_A^{h\alpha}{}_{|\alpha} - Q_A^{vi}{}_{\|i} = 0, \tag{4.35}$$

$$\frac{\partial \mathcal{L}}{\partial H_\alpha^a} - H_a^{h\alpha\beta}{}_{|\beta} - H_a^{v\alpha i}{}_{\|i} = 0, \tag{4.36}$$

$$\frac{\partial \mathcal{L}}{\partial V_i^a} - V_a^{hi\alpha}{}_{|\alpha} - V_a^{vij}{}_{\|j} = 0. \tag{4.37}$$

(ii) *The conservation laws are given by*

$$J_a^{h\beta}{}_{|\beta} + J_a^{vj}{}_{\|j} = 0. \tag{4.38}$$

We have to mention that here the horizontal and vertical covariant derivatives are also taken with respect to the Vrănceanu connection. Thus this is another proof of the usefulness of the Vrănceanu connection in applying the geometry of foliations in physics.

6.5 Bianchi Identities for Strength Fields

In the first part of this section we show that the horizontal and vertical gauge covariant derivatives of strength fields are adapted tensor fields. Then by using the Vrănceanu connection we obtain the Bianchi identities for strength fields and their gauge covariant derivatives.

Let ∇ be an adapted linear connection on the total space E of a vector bundle $\xi = (E, \pi, M)$ endowed with a horizontal distribution HE. According to (1.18) and (1.19), ∇ is locally given by the functions $(F_\alpha{}^\gamma{}_\beta, L_\alpha{}^\gamma{}_i, D_i{}^k{}_\alpha, C_i{}^k{}_j)$. Then we define the **horizontal gauge covariant derivatives** and the **vertical gauge covariant derivative** of the strength fields by

$$\begin{aligned}
&\text{(a) } R^a{}_{\alpha\beta|\gamma} = \frac{\delta R^a{}_{\alpha\beta}}{\delta x^\gamma} + C_b{}^a{}_c R^b{}_{\alpha\beta} H^c_\gamma \\
&\qquad - R^a{}_{\varepsilon\beta} F_\alpha{}^\varepsilon{}_\gamma - R^a{}_{\alpha\varepsilon} F_\beta{}^\varepsilon{}_\gamma, \\
&\text{(b) } P^a{}_{\alpha i|\beta} = \frac{\delta P^a{}_{\alpha i}}{\delta x^\beta} + C_b{}^a{}_c P^b{}_{\alpha i} H^c_\beta \\
&\qquad - P^a{}_{\varepsilon i} F_\alpha{}^\varepsilon{}_\beta - P^a{}_{\alpha j} D_i{}^j{}_\beta, \\
&\text{(c) } S^a{}_{ij|\alpha} = \frac{\delta S^a{}_{ij}}{\delta x^\alpha} + C_b{}^a{}_c S^b{}_{ij} H^c_\alpha \\
&\qquad - S^a{}_{kj} D_i{}^k{}_\alpha - S^a{}_{ik} D_j{}^k{}_\alpha,
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
&\text{(a) } R^a{}_{\alpha\beta\|i} = \frac{\partial R^a{}_{\alpha\beta}}{\partial t^i} + C_b{}^a{}_c R^b{}_{\alpha\beta} V^c_i \\
&\qquad - R^a{}_{\varepsilon\beta} L_\alpha{}^\varepsilon{}_i - R^a{}_{\alpha\varepsilon} L_\beta{}^\varepsilon{}_i, \\
&\text{(b) } P^a{}_{\alpha i\|j} = \frac{\partial P^a{}_{\alpha i}}{\partial t^j} + C_b{}^a{}_c P^b{}_{\alpha i} V^c_j \\
&\qquad - P^a{}_{\varepsilon i} L_\alpha{}^\varepsilon{}_j - P^a{}_{\alpha k} C_i{}^k{}_j, \\
&\text{(c) } S^a{}_{ij\|k} = \frac{\partial S^a{}_{ij}}{\partial t^k} + C_b{}^a{}_c S^b{}_{ij} V^c_k \\
&\qquad - S^a{}_{hj} C_i{}^h{}_k - S^a{}_{ih} C_j{}^h{}_k,
\end{aligned} \tag{5.2}$$

respectively. Now we state the following.

Proposition 5.1. *The horizontal and vertical gauge covariant derivatives* ${R^a}_{\alpha\beta|\gamma}$, ${P^a}_{\alpha i|\beta}$, ${S^a}_{ij|\alpha}$, ${R^a}_{\alpha\beta\|i}$, ${P^a}_{\alpha i\|j}$ *and* ${S^a}_{ij\|k}$ *are the local components of some adapted tensor fields on* E *of type* $(0,0;0,3)$, $(0,1;0,2)$, $(0,2;0,1)$, $(0,1;0,2)$, $(0,2;0,1)$ *and* $(0,3;0,0)$ *respectively for each* $a \in \{1,...,r\}$.

Proof. Since ${R^a}_{\alpha\beta}$ are the components of an adapted tensor field of type $(0,0;0,2)$ we have

$$
{R^a}_{\alpha\beta} = {\widetilde{R}^a}_{\nu\mu} J^\nu_\alpha J^\mu_\beta, \tag{5.3}
$$

with respect to the coordinate transformations (1.1). Then we apply $\dfrac{\delta}{\delta x^\gamma}$ to (5.3) and by using (1.7) we obtain

$$
\frac{\delta {R^a}_{\alpha\beta}}{\delta x^\gamma} = \frac{\delta {\widetilde{R}^a}_{\nu\mu}}{\delta \widetilde{x}^\varepsilon} J^\nu_\alpha J^\mu_\beta J^\varepsilon_\gamma + {\widetilde{R}^a}_{\nu\mu} J^\mu_\beta \frac{\partial^2 \widetilde{x}^\nu}{\partial x^\gamma \partial x^\alpha} + {\widetilde{R}^a}_{\nu\mu} J^\nu_\alpha \frac{\partial^2 \widetilde{x}^\mu}{\partial x^\gamma \partial x^\beta} \cdot \tag{5.4}
$$

Also, from (2.3.9) we deduce that

$$
{F_\alpha}^\gamma{}_\beta J^\varepsilon_\gamma = {\widetilde{F}_\mu}{}^\varepsilon{}_\nu J^\mu_\alpha J^\nu_\beta + \frac{\partial^2 \widetilde{x}^\varepsilon}{\partial x^\alpha \partial x^\beta} \cdot \tag{5.5}
$$

Then, by direct calculations using (5.3)–(5.5), (1.13a) and (5.1a) for ${\widetilde{R}^a}_{\nu\mu}$, we obtain

$$
{\widetilde{R}^a}_{\nu\mu|\varepsilon} J^\nu_\alpha J^\mu_\beta J^\varepsilon_\gamma = {R^a}_{\alpha\beta|\gamma},
$$

that is, ${R^a}_{\alpha\beta|\gamma}$ define an adapted tensor field of type $(0,0;0,3)$ for each $a \in \{1,...,r\}$. Next, by (2.3.10) we deduce that ${L_\alpha}^\gamma{}_i$ from (1.18b) define an adapted tensor field, i.e., we have

$$
{L_\alpha}^\gamma{}_i J^\varepsilon_\gamma = {\widetilde{L}_\beta}{}^\varepsilon{}_j J^\beta_\alpha B^j_i. \tag{5.6}
$$

Now, we take partial derivatives in (5.3) with respect to t^i and by using (1.2a) we obtain

$$
\frac{\partial {R^a}_{\alpha\beta}}{\partial t^i} = \frac{\partial {\widetilde{R}^a}_{\nu\mu}}{\partial \widetilde{t}^j} J^\nu_\alpha J^\mu_\beta B^j_i. \tag{5.7}
$$

Then, by using (5.3), (5.6), (5.7), (1.13b) and (5.2a) we deduce that

$$
{\widetilde{R}^a}_{\nu\mu\|j} J^\nu_\alpha J^\mu_\beta B^j_i = {R^a}_{\alpha\beta\|i},
$$

which means that ${R^a}_{\alpha\beta\|i}$ define an adapted tensor field of type $(0,1;0,2)$ for each $a \in \{1,...,r\}$. Therefore, both the horizontal and vertical gauge covariant derivatives of horizontal strength fields define adapted tensor fields on E. In a similar way can prove the same assertion for mixed and vertical strength fields. ∎

Proposition 5.2. *The local gauge action of the Lie group G on the horizontal and vertical gauge covariant derivatives of strength fields is given by the adjoint representation of G, that is, we have:*

$$\begin{aligned}
&\text{(a) } \overset{*}{\delta} R^a{}_{\alpha\beta|\gamma} = \varepsilon^b C_b{}^a{}_c R^c{}_{\alpha\beta|\gamma},\\
&\text{(b) } \overset{*}{\delta} P^a{}_{\alpha i|\beta} = \varepsilon^b C_b{}^a{}_c P^c{}_{\alpha i|\beta},\\
&\text{(c) } \overset{*}{\delta} S^a{}_{ij|\alpha} = \varepsilon^b C_b{}^a{}_c S^c{}_{ij|\alpha},
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
&\text{(a) } \overset{*}{\delta} R^a{}_{\alpha\beta\|i} = \varepsilon^b C_b{}^a{}_c R^c{}_{\alpha\beta\|i},\\
&\text{(b) } \overset{*}{\delta} P^a{}_{\alpha i\|j} = \varepsilon^b C_b{}^a{}_c P^c{}_{\alpha i\|j},\\
&\text{(c) } \overset{*}{\delta} S^a{}_{ij\|k} = \varepsilon^b C_b{}^a{}_c S^c{}_{ij\|k}.
\end{aligned} \tag{5.9}$$

Proof. Apply $\overset{*}{\delta}$ to (5.1a) and by using (3.39), (3.6) and (3.9) we obtain (5.8a). The other equalities are obtained in a similar way. ■

Next, we consider the semi–holonomic frame field $\left\{ \frac{\partial}{\partial t^i}, \frac{\delta}{\delta x^\alpha} \right\}$ on E and write down the following Jacobi identities

$$\sum_{(\alpha,\beta,\gamma)} \left\{ \left[\left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta} \right], \frac{\delta}{\delta x^\gamma} \right] \right\} = 0, \tag{5.10}$$

and

$$\begin{aligned}
&\left[\left[\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta} \right], \frac{\partial}{\partial t^i} \right] + \left[\left[\frac{\delta}{\delta x^\beta}, \frac{\partial}{\partial t^i} \right], \frac{\delta}{\delta x^\alpha} \right]\\
&\qquad + \left[\left[\frac{\partial}{\partial t^i}, \frac{\delta}{\delta x^\alpha} \right], \frac{\delta}{\delta x^\beta} \right] = 0.
\end{aligned} \tag{5.11}$$

Then, by using (1.16) and (1.35), it is easy to see that (5.10) and (5.11) become

$$\sum_{(\alpha,\beta,\gamma)} \left\{ \frac{\delta T_\alpha{}^i{}_\beta}{\delta x^\gamma} + T_\alpha{}^j{}_\beta D_j{}^i{}_\gamma \right\} = 0, \tag{5.12}$$

and

$$\frac{\partial T_\alpha{}^j{}_\beta}{\partial t^i} = \frac{\delta D_i{}^j{}_\alpha}{\delta x^\beta} - \frac{\delta D_i{}^j{}_\beta}{\delta x^\alpha} + D_i{}^k{}_\alpha D_k{}^j{}_\beta - D_i{}^k{}_\beta D_k{}^j{}_\alpha, \tag{5.13}$$

respectively. Now, we can prove the following.

Theorem 5.3. *The horizontal and vertical gauge covariant derivatives of the strength fields with respect to the Vrănceanu connection satisfy the following identities:*

$$\sum_{(\alpha,\beta,\gamma)} \left\{R^a{}_{\alpha\beta|\gamma} + P^a{}_{\alpha i} T_\beta{}^i{}_\gamma\right\} = 0, \tag{5.14}$$

$$\sum_{(i,j,k)} \left\{S^a{}_{ij\|k}\right\} = 0, \tag{5.15}$$

$$P^a{}_{\alpha i\|j} - P^a{}_{\alpha j\|i} + S^a{}_{ij|\alpha} = 0, \tag{5.16}$$

$$P^a{}_{\alpha i|\beta} - P^a{}_{\beta i|\alpha} - R^a{}_{\alpha\beta\|i} - S^a{}_{ij} T_\alpha{}^j{}_\beta = 0. \tag{5.17}$$

Proof. First, by using (5.1a), (3.19), (1.16), (3.9) and taking into account that $F_\alpha{}^\gamma{}_\beta = F_\beta{}^\gamma{}_\alpha$, we obtain

$$\begin{aligned} \sum_{(\alpha,\beta,\gamma)} \left\{R^a{}_{\alpha\beta|\gamma}\right\} &= \sum_{(\alpha,\beta,\gamma)} \left\{\frac{\delta R^a{}_{\alpha\beta}}{\delta x^\gamma} + C_b{}^a{}_c R^b{}_{\alpha\beta} H^c_\gamma\right\} \\ &= \sum_{(\alpha,\beta,\gamma)} \left\{\frac{\delta T_\alpha{}^i{}_\beta}{\delta x^\gamma} V^a_i - \frac{\partial H^a_\alpha}{\partial t^i} T_\beta{}^i{}_\gamma + \frac{\delta V^a_i}{\delta x^\gamma} T_\alpha{}^i{}_\beta + C_b{}^a{}_c H^c_\gamma T_\alpha{}^i{}_\beta V^b_i\right\}. \end{aligned} \tag{5.18}$$

On the other hand, (3.20) implies

$$\begin{aligned} &\sum_{(\alpha,\beta,\gamma)} \left\{P^a{}_{\alpha i} T_\beta{}^i{}_\gamma\right\} \\ &= \sum_{(\alpha,\beta,\gamma)} \left\{\frac{\partial H^a_\alpha}{\partial t^i} T_\beta{}^i{}_\gamma - \frac{\delta V^a_i}{\delta x^\alpha} T_\beta{}^i{}_\gamma - C_b{}^a{}_c H^c_\gamma T_\alpha{}^i{}_\beta V^b_i + T_\alpha{}^j{}_\beta D_j{}^i{}_\gamma V^a_i\right\}. \end{aligned} \tag{5.19}$$

Then (5.14) follows from (5.18) and (5.19) via (5.12). Next, by using (5.2c) and taking into account that $C_j{}^i{}_k = C_k{}^i{}_j$, we deduce that

$$\sum_{(i,j,k)} \left\{S^a{}_{ij|k}\right\} = \sum_{(i,j,k)} \left\{\frac{\partial S^a{}_{ij}}{\partial t^k} + C_b{}^a{}_c S^b{}_{ij} V^c_k\right\}. \tag{5.20}$$

Now we use (3.21) and (3.9a) and obtain

$$\begin{aligned} &\sum_{(i,j,k)} \left\{\frac{\partial S^a{}_{ij}}{\partial t^k}\right\} \\ &= \sum_{(i,j,k)} \left\{\frac{\partial^2 V^a_i}{\partial t^k \partial t^j} - \frac{\partial^2 V^a_j}{\partial t^k \partial t^i} - C_b{}^a{}_c \frac{\partial V^c_i}{\partial t^k} V^b_j - C_b{}^a{}_c V^c_i \frac{\partial V^b_j}{\partial t^k}\right\} \\ &= C_c{}^a{}_b \sum_{(i,j,k)} \left\{\frac{\partial V^c_i}{\partial t^k} V^b_j + V^c_i \frac{\partial V^b_j}{\partial t^k}\right\}. \end{aligned} \tag{5.21}$$

By using (3.21) and (3.9b), we infer that

$$\sum_{(i,j,k)} \left\{ C_b{}^a{}_c S^b{}_{ij} V_k^c \right\}$$

$$= C_b{}^a{}_c \sum_{(i,j,k)} \left\{ \frac{\partial V_i^b}{\partial t^j} - \frac{\partial V_j^b}{\partial t^i} \right\} V_k^c + V_i^e V_j^d V_k^c \sum_{(c,d,e)} C_c{}^a{}_b C_d{}^b{}_e \tag{5.22}$$

$$= C_b{}^a{}_c \sum_{(i,j,k)} \left\{ V_i^c \frac{\partial V_j^b}{\partial t^k} + V_k^b \frac{\partial V_j^c}{\partial t^i} \right\}.$$

Thus (5.15) follows from (5.20)–(5.22). By a little longer calculation than above, using (3.19)–(3.21), (5.1), (5.2), (3.9), (1.16), (1.35) and (5.13), we obtain (5.16) and (5.17). ∎

We call (5.14)–(5.17) the **Bianchi identities** for the strength fields $R^a{}_{\alpha\beta}$, $P^a{}_{\alpha i}$ and $S^a{}_{ij}$ with respect to the Vrănceanu connection. The Bianchi identities with respect to an arbitrary adapted connection have been obtained by Bejancu [B89]. In particular, when E is a trivial vector bundle $M \times \mathbb{R}^n$, the above Bianchi identities become

$$\sum_{(\alpha,\beta,\gamma)} \left\{ R^a{}_{\alpha\beta|\gamma} \right\} = 0, \tag{5.23}$$

$$\sum_{(i,j,k)} \left\{ S^a{}_{ij\|k} \right\} = 0, \tag{5.24}$$

$$P^a{}_{\alpha i\|j} - P^a{}_{\alpha j\|i} + S^a{}_{ij|\alpha} = 0, \tag{5.25}$$

$$P^a{}_{\alpha i|\beta} - P^a{}_{\beta i|\alpha} - R^a{}_{\alpha\beta\|i} = 0, \tag{5.26}$$

since in this case the Vrănceanu connection is torsion–free.

BASIC NOTATIONS AND TERMINOLOGY

Throughout the book we use the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their range.

All manifolds are supposed to be connected, paracompact and smooth (differentiable of class C^∞). Also, all geometric objects on manifolds are supposed to be smooth.

The quotations of formulas, theorems, etc., are made as follows: Formula (1.2.3), Theorem 1.2.3, Proposition 1.2.3, Lemma 1.2.3, Corollary 1.2.3, Remark 1.2.3 or Example 1.2.3, means that they have the number 2.3 in Chapter 1. When we do not mention the first number, it is understood that we refer to a formula, theorem, etc., in the chapter where the quotation is made. Thus Theorem 2.3 means the theorem with the number 2.3 in the chapter where we make the quotation. The sections are quoted as they are in the chapter. Thus Section 1.3 means the third section in Chapter 1.

We now present the basic notations and symbols which appear frequently throughout the book.

$\mathbb{R}^m$ – the space of m–tuples $(x^1, ..., x^m)$ of real numbers

M – an m–dimensional smooth manifold

TM – tangent bundle of M

T_xM – tangent space of M at x

T^*M – cotangent bundle of M

T_x^*M – cotangent space of M at x

$\Pi_1(M)$ – the fundamental group of M

$\mathcal{D}$ – a distribution on a manifold

$\mathcal{D}_x$ – the fiber of $\mathcal{D}$ over $x \in M$

$\mathcal{F}$ – a foliation on a manifold

g or $\widetilde{g}$ – a semi–Riemannian (Riemannian) metric on a manifold

$(M, \mathcal{F})$ – a foliated manifold

$(M, g, \mathcal{F})$ – a foliated semi–Riemannian (Riemannian) manifold

$F(M)$ – the algebra of smooth functions on M

$\varGamma(\mathcal{D})$ – the $F(M)$–module of smooth sections of $\mathcal{D}$ (this notation is also used for any other vector bundle over M)

$L_x(\mathcal{D}_x, \mathcal{D}_x)$ – the vector space of linear mappings on $\mathcal{D}_x$

$L(\mathcal{D}, \mathcal{D})$ – the vector bundle with fibers $L_x(\mathcal{D}_x, \mathcal{D}_x)$

$\varGamma(\mathcal{D})^r$ – $\underbrace{\varGamma(\mathcal{D})\times\cdots\times\varGamma(\mathcal{D})}_{r \text{ times}}$

$\nabla, \widetilde{\nabla}$ – linear connections on a manifold or on a vector bundle. If $\widetilde{\nabla}$ is the Levi–Civita connection on (M, g), then we denote by ∇° and ∇^{*} the Schouten–Van Kampen connection and the Vrănceanu connection respectively defined by $\widetilde{\nabla}$

∇ and $\nabla^{\perp}$ are the induced connections by $\widetilde{\nabla}$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively

D and $D^{\perp}$ are the intrinsic connections on $\mathcal{D}$ and$\mathcal{D}^{\perp}$ respectively

If $(x^i, x^{\alpha}),\ i \in \{1, ..., n\},\ \alpha \in \{n+1, ..., n+p\}$ are the local coordinates on a foliated manifold $(M, \mathcal{F})$, where (x^i) are the leaf coordinates, then

$$\frac{\delta}{\delta x^{\alpha}} = \frac{\partial}{\partial x^{\alpha}} - A^i_{\alpha}\,\frac{\partial}{\partial x^i}\,,\ \alpha \in \{n+1, ..., n+p\}$$

determine locally the transversal distribution of $\mathcal{F}$

$\left\{\dfrac{\partial}{\partial x^i}\,,\dfrac{\delta}{\delta x^{\alpha}}\right\}$ is the semi–holonomic frame field on $(M, \mathcal{F})$ or $(M, g, \mathcal{F})$

The structural and transversal covariant derivatives of an adapted tensor field $T = \left(T^{i\alpha}_{j\beta}\right)$ with respect to an adapted linear connection on $(M, \mathcal{F})$ are denoted by $T^{i\alpha}_{j\beta\|k}$ and $T^{i\alpha}_{j\beta|\gamma}$ respectively.

$\sum\limits_{(i,j,k)}$ and $\sum\limits_{(\alpha,\beta,\gamma)}$ – cyclic sums with respect to the indices (i, j, k) and (α, β, γ) respectively

$\sum\limits_{(X,Y,Z)}$ – cyclic sum with respect to the vector fields (X, Y, Z)

References

[Abe73] Abe, K.: Applications of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions, Tôhoku Math. J., **25**, 425–444 (1973).

[Abe85] Abe, N.: General connections on vector bundles, Kodai Math. J., **8**, 322–329 (1985).

[Art75] Artin, E.: Geometric Algebra, Interscience Publishers, New York (1975).

[Asa85] Asanov, G.S.:Finsler Geometry, Relativity and Gauge Theories, D. Reidel Publish. Comp., Dordrecht (1985).

[AM55] Auslander, L. and Marcus, L.: Holonomy of flat affinely connected manifolds, Annals of Math., **62** (1), 139–151 (1955).

[BBI98] Băditoiu, G., Buchner, K. and Ianuş, S.: Some remarkable connections and semi–Riemannian submersions, Bull. Math. Soc. Sci. Math. Roumanie, **48** (89), No. 3, 153–169 (1998).

[BCU81] Barros, M., Chen, B.Y. and Urbano, F.: Quaternion CR–submanifolds of quaternion manifolds, Kodai Math. J., **4**, 399–418 (1981).

[B78] Bejancu, A.: CR–Submanifolds of a Kähler manifold, I, Proc. Amer. Math. Soc., **69**, 134–142 (1978).

[B86a] Bejancu, A.: Geometry of CR–Submanifolds, D. Reidel Publ. Comp., Dordrecht (1986).

[B86b] Bejancu, A.: QR–Submanifolds of quaternion Kählerian manifolds, Chinese J. Math., **14**, No. 2, 81–94 (1986).

[B88] Bejancu, A.: Foundations of direction–dependent gauge theories, Mechanics Seminar Timişoara (Romania), No. 13, 1988, 60 pages.

[B89] Bejancu, A.: Generalized gauge theory, Colloquia Math. Soc. J. Bolyai, **56**, Differential Geometry, Eger, 101–126 (1989).

[B96] Bejancu, A.: Geometry of degenerate hypersurfaces, Arab J. Math. Sc., **2**, 1–38 (1996).

[BF00a] Bejancu, A. and Farran, H.R.: Geometry of Pseudo–Finsler Submanifolds, Kluwer Acad. Publ., Dordrecht (2000).

[BF00b] Bejancu, A. and Farran, H.R.: A geometric characterization of Finsler manifolds of constant curvature $k = 1$, Internat. J. Math. and Math. Sci., **23** (7), 1–9 (2000).

[BF02] Bejancu, A. and Farran, H.R.: Finsler metrics of positive constant flag curvature on Sasakian space forms, Hokkaido Math. J., **31** (2), 459–468 (2002).

[BF03a] Bejancu, A. and Farran, H.R.: Structural and transversal geometry of foliations, Internat. J. Pure and Appl. Math., **9**, No. 4, 419–450 (2003).

[BF03b] Bejancu, A. and Farran, H.R.: Lightlike foliations of codimension one, Publ. Math. Debrecen, **62**, f. 3-4, 325–336 (2003).

[BF03c] Bejancu, A. and Farran, H.R.: Randers manifolds of positive constant curvature, Internat. J. Math. and Math. Sci., **26** (18), 1155–1165 (2003).

[BF05] Bejancu, A. and Farran, H.R.: On the geometry of semi–Riemannian distributions, to appear in Anal. Univ. "Al.I. Cuza" Iaşi.

[BO87] Bejancu, A. and Otsuki, T.: General Finsler connections on a Finsler vector bundle, Kodai Math. J., **10**, 143–152 (1987).

[BP81] Bejancu, A. and Papaghiuc, N.: Semi–invariant submanifolds of a Sasakian manifold, An. St. Univ. "Al.I. Cuza" Iaşi, **27**, 163–170 (1981).

[Ber60] Bernard, D.: Sur la géométrie différentielle des G–structures, Ann. Inst. Fourier (Grenoble), **10**, 151–270 (1960).

[Ber01] Berndt, R.: An Introduction to Symplectic Geometry, Amer. Math. Soc., Providence, Graduate Studies in Math., Vol. **26** (2001).

[Be87] Besse, A.L.: Einstein Manifolds, Springer–Verlag, Berlin (1987).

[Bla76] Blair, D.E.: Contact Manifolds in Riemannian Geometry, Lecture Notes in Math., **509**, Springer–Verlag, Berlin (1976).

[Bla01] Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds, Birkhäuser, Basel (2001).

[BC79] Blair, D.E. and Chen, B.Y.: On CR–submanifolds of Hermitian manifolds, Israel J. Math., **34**, 353–363 (1979).

[BKP95] Blair, D.E., Koufogiorgos, T., Papantoniou, B.J.: Contact metric manifolds satisfying a nullity condition, Israel J. Math., **91**, 189–214 (1995).

[BC70] Brickell, F. and Clark, R.S.: Differentiable Manifolds, An Introduction, Van Nostrand, New York, (1970).

[BW86] Brito, G.B. and Walczak, P.G.: Totally geodesic foliations with integrable normal bundles, Bol. Soc. Bras. Mat., **17**, 41–46 (1986).

[Cai90] Cairns, G.: Totally umbilic Riemannian foliations, Michigan Math. J., **37**, 145-159 (1990).

[CN85] Camacho, C. and Neto, A.L.: Geometric Theory of Foliations, Progress in Math., Birkhäuser, Boston (1985).

[CC03] Candel, A. and Conlon, L.: Foliations, I and II, Graduate Studies in Math., Vol. **23** and Vol. **60**, Amer. Math. Soc., Providence, Rhode Island (2000) and (2003).

[Car81] Carrière, Y.: Flots riemanniens et feuilletages géodésibles de codimension un, Thèse Université des Sciences et Techniques de Lille, (1981).

[CN84] Chaichian, M. and Nelipa, N.F.: Introduction to Gauge Field Theories, Springer–Verlag, Berlin (1984).

[C73] Chen, B.Y.: Geometry of Submanifolds, Marcel Dekker, New York (1973).

[C81] Chen, B.Y.: CR–Submanifolds of a Kähler manifold, I., J. Differential Geometry, **16**, 305–323 (1981).

[Che48] Chern, S.S.: Local equivalence and Euclidean connection in Finsler spaces, Sci. Rep. Tsing Hua Univ., **5**, 95–121 (1948).

[Che53] Chern, S.S.: Pseudo–groupes continus infinis, Colloque de Géométrie Différentielle, Strasbourg, 119–136 (1953).

[Che66] Chern, S.S.: The geometry of G–structures, Bull. Amer. Math. Soc., **72**, 167–219 (1966).

[Cor85] Cordero, L.A.: Sheaves and cohomologies associated to subfoliations, Resultate Math., **8**, 9–20 (1985).
[CG76] Cordero, L.A. and Gadea, P.M.: Exotic characteristic classes and subfoliations, Ann. Inst. Fourier, Grenoble, **26**, 225–237 (1976).
[CFG96] Cruceanu, V., Fortuny, P. and Gadea, P.M.: A survey on paracomplex geometry, Rocky Mountain J. Math., **28**, 83–115 (1996).
[CB88] Currás–Bosch, C.: The geometry of totally geodesic foliations admitting Killing fields, Tôhoku Math. J., **40**, 535–548 (1988).
[deR52] de Rham, G.: Sur la réductibilité d'un espace de Riemann, Comment. Math. Helv., **26**, 328–344 (1952).
[ER44] Ehresmann, Ch. and Reeb, G.: Sur les champs d'éléments de contact de dimension p complètement intégrables dans une variété continuement différentiable V_n, C.R. Acad. Sci. Paris, **216**, 628–630 (1944).
[Esc82] Escobales, Jr., R.H.: The integrability tensor for bundle–like foliations, Trans. Amer. Math. Soc., **270**, No. 1, 333–339 (1982).
[Far79] Farran, H.R.: Foliated pseudoriemannian and symplectic manifolds, Progress of Math., **13**, 59–64 (1979).
[Far80] Farran, H.R.: G–structures on manifolds with parallel foliations, J.Kuwait Univ. (Science), **7**, 59–68 (1980).
[Far81] Farran, H.R.: Foliations by suspensions, Proc.Conference on Algebra and Geometry, Kuwait Univ., 83–87 (1981).
[Far83] Farran, H.R.: Almost product Riemannian manifolds, Czechoslovak Math.J., **33**, 119–125 (1983).
[FR96] Farran, H.R. and Robertson, S.A.: Symmetric foliations, Textos de Matematica, Series B, No. **10**, 12–23 (1996).
[Fei75] Feigin, B.L.: Characteristic classes of flags of foliations, Functional Anal. Appl., **9**, 312–317 (1975).
[Fer70] Ferus, D.: Totally geodesic foliations, Math. Ann., **188**, 313–316 (1970).
[Fuj60] Fujimoto, A.: On the structure tensor of G–structure, Mem. Coll. Sci. Univ. Kyoto, Ser, A., **18**, 157–169 (1960).
[Fur72] Furness, P.M.D.: Parallel foliations, Ph.D. Thesis, University of Durham (1972).
[Fur74] Furness, P.M.D.: Affine foliations of codimension one, Quart. J. Math. Oxford, **25** (2), 151–161 (1974).
[Ghy83] Ghys, E.: Classification des feuilletages totalement géodésiques de codimension un, Comment. Math. Helv., **58**, 543–572 (1983).
[Gra67] Gray, A.: Pseudo–Riemannian almost product manifolds and submersions, J. Math. Mech., **16**, 715–737 (1967).
[GG85] Gromoll, D. and Grove, K.: One-dimensional metric foliations in constant curvature spaces, Differential Geometry and Complex Analysis, Springer–Verlag, New York, 165–168 (1985).
[Hae80] Haefliger, A.: Some remarks on foliations with minimal leaves, J. Differential Geom., **15**, 269–284 (1980).
[HH83] Hector, G. and Hirsch, U.: Introduction to the Geometry of Foliations, Parts A and B, Vieweg, Braunschweig (1981) and (1983).
[Hel01] Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces, Amer. Math. Soc., Providence, Graduate Studies in Math., Vol. **34** (2001).
[Hor27] Horak, Z.: Sur les systèmes non holonomes, Bull. Internat. Acad. Sci. Bohème, 1–18 (1927).

[Ian71] Ianuş, S.: Some almost product structures on manifolds with linear connections, Kodai Math. Sem. Rep., **23**, 305–310 (1971).

[Jay92] Jayne, N.: Legendre Foliations on Contact Metric Manifolds, Ph.D. Thesis, Massey University (1992).

[Jay94] Jayne, N.: A note on the sectional curvature of Legendre foliations, Yokohama Math. J., **41**, 153–161 (1994).

[JW80] Johnson, D.L. and Whitt, L.B.: Totally geodesic foliations, J. Differential Geometry, **15**, 225–235 (1980).

[KT71] Kamber, F.W. and Tondeur, Ph.: Invariant differential operators and the cohomology of Lie algebra sheaves, Memoirs Amer. Math. Soc., **113**, 1–125 (1971).

[KT82] Kamber, F.W. and Tondeur, Ph.: Harmonic foliations, Proc. Nat. Sci. Foundation Conference on Harmonic Maps (Tulane, Dec. 1980), Lecture Notes in Math., **949**, Springer–Verlag, New York, 87–121 (1982).

[Kas59] Kashiwabara, S.: The decomposition of a differentiable manifold and its applications, Tôhoku Math. J., **11**, 43–53 (1959).

[KT92] Kim, H. and Tondeur, Ph.: Riemannian foliations on manifolds with nonnegative curvature, Manuscripta Math., **74**, 39–45 (1992).

[Kit86] Kitahara, H.: Differential Geometry of Riemannian Foliations, Lecture Notes, Kyungpook National Univ. (1986).

[KN63] Kobayashi, S. and Nomizu, K., Foundations of Differential Geometry, Vol. I, Interscience, New York (1963).

[K90] Koike, N.: Totally umbilic foliations and decomposition theorems, Saitama Math. J., **8**, 1–18 (1990).

[Kow80] Kowalski, O.: Generalized Symmetric Spaces, Lecture Notes in Math., **805**, Springer–Verlag, Berlin, 1980.

[Law74] Lawson, Jr., H.B.: Foliations, Bull.Amer. Math. Soc., **80**, No. 3, 369-418 (1974).

[Lib91] Libermann, P.: Legendre foliations on contact manifolds, Diff. Geometry and Its Appl., **1**, 57–76 (1991).

[Mat86] Matsumoto, M.: Foundations of Finsler Geometry and Special Finsler Spaces, Kaiseisha Press, Otsushi, Shigaken (1986).

[Mir82] Miron, R.: Vector bundles Finsler geometry, Proc. Nat. Sem. Finsler Spaces, Braşov, 147–188 (1982).

[Mol88] Molino, P.: Riemannian Foliations, Progress in Math., Vol. **73**, Birkhäuser, Boston (1988).

[Nem85] Nemoto, H.: On differential geometry of general connections, TRU Math., **21**, 67–94 (1985).

[NN57] Newlander, A. and Nirenberg, L.: Complex analytic coordinates in almost complex manifolds, Ann. of Math., **65**, 391–404 (1957).

[Nit71] Nitecki, Z.: Differentiable Dynamics, M.I.T., Cambridge (1971).

[Oku62] Okumura, M.: On infinitesimal conformal and projective transformations of normal contact spaces, Tôhoku Math. J., **14**, 398–412 (1962).

[O66] O'Neill, B.: The fundamental equations of a submersion, Michigan Math. J., **13**, 459–469 (1966).

[O83] O'Neill, B.: Semi–Riemannian Geometry with Applications to Relativity, Academic Press, New York (1983).

[Orn86] Ornea, L.: On CR–submanifolds of a locally conformal Kähler manifold, Demonstratio Mathematica, **19**, No. 4, 863–869 (1986).

[Osh83] Oshikiri, G.: Totally geodesic foliations and Killing fields, Tôhoku Math. J., **35**, 387–392 (1983).

[Osh86] Oshikiri, G.: Totally geodesic foliations and Killing fields, II, Tôhoku Math. J., **38**, 351–356 (1986).

[Osh90] Oshikiri, G.: Mean curvature functions of codimension–one foliations, Comment. Math. Helv., **65**, 79–84 (1990).

[Osh91] Oshikiri, G.: Mean curvature functions of codimension-one foliations, Comment. Math. Helv., **66**, 512–520 (1991).

[Ots61] Otsuki, T.: On general connections, I; II, Math. J. Okayama Univ., **9**, 99–164 (1960); Math. J. Okayama Univ., **10**, 113–124 (1961).

[Pan90] Pang, M.Y.: The structure of Legendre foliations, Trans. Amer. Math. Soc., **320** (2), 417–455 (1990).

[PW52] Patterson, E.M. and Walker, A.G.: Riemann Extensions, Quart. J. Math., **3**, 19–28 (1952).

[Rei59a] Reinhart, B.L.: Foliated manifolds with bundle–like metric, Annals of Math., **69** (2), 119–132 (1959).

[Rei59b] Reinhart, B.L.: Harmonic integrals on foliated manifolds, Amer. J. Math., **81**, 529–536 (1959).

[Rei83] Reinhart, B.L.: Differential Geometry of Foliations, Springer–Verlag, Berlin (1983).

[Rob70] Robertson, S.A.: Grid Manifolds, J. Differential Geometry, **4**, 245–253 (1970).

[RF74] Robertson, S.A. and Furness, P.M.D.: Parallel framings and foliations on pseudoriemannian manifolds, J. Differential Geometry, **9**, 409–422 (1974).

[Rov98] Rovenskii, V.: Foliations on Riemannian Manifolds and Submanifolds, Birkhäuser, Boston (1998).

[Rum79] Rummler, H.: Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts, Comment. Math. Helv., **54**, 224–239 (1979).

[San82] Sanini, A.: Foliazioni minimali e totalmente geodetiche, Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Natur., **116**, 117–126 (1982).

[Sch28] Schouten, J.A.: On non–holonomic connexions, Proc. Kon. Akad. Amsterdam, **31**, 291–299 (1928).

[Sch54] Schouten, J.A.: Ricci Calculus, 2^{nd} edition, Springer–Verlag, Berlin (1954).

[SS21] Schouten, J.A. and Struik, D.J.: On some properties of general manifolds related to Einstein's theory of gravitation, Amer. J. Math., **43**, 213-216 (1921).

[SVK30] Schouten, J.A. and Van Kampen, E.R.: Zur Einbettungs – und Krümmungstheorie nichtholonomer Gebilde, Math. Ann., **103**, 752–783 (1930).

[Shu69] Shub, M.: Endomorphisms of compact differentiable manifolds, Amer. J. Math., **91**, 175–199 (1969).

[Stee51] Steenrod, N.: Topology of Fibre Bundles, Princeton Univ. Press, Princeton, New Jersey (1951).

[Ste83] Sternberg, S.: Lectures on Differential Geometry, 2^{nd} edition, Chelsea Publ. Comp., New York (1983).

[Sul79] Sullivan, D.: A homological characterization of foliations consisting of minimal surfaces, Comment. Math. Helv., **54**, 218–223 (1979).

[Tam76] Tamura, I.: Topology of Foliations: An Introduction, AMS Transactions of Mathematical Monographs, **97** (1992) (first published in Japanese by Iwanami Shoten Publ., Tokyo (1976)).
[Tan68] Tanno, S.: The topology of contact Riemannian manifolds, Illinois J. Math., **12**, 700–717 (1968).
[Tan69] Tanno, S.: Sasakian manifolds with constant φ–holomorphic sectional curvature, Tôhoku Math. J., **21**, 501–507 (1969).
[Tan72] Tanno, S.: A theorem on totally geodesic foliations and its applications, Tensor, N.S., **24**, 116–122 (1972).
[Tash69] Tashiro, Y.: On contact structures of tangent spheres bundles, Tôhoku Math. J., **21**, 117–143 (1969).
[Tho39] Thomas, T.Y.: The decomposition of Riemann spaces in the large, Monatsh. Math. Phys., **47**, 388–418 (1939).
[Thu76] Thurston, W.: Some simple examples of symplectic manifolds, Proc. Amer. Math. Soc., **55**, 467–468 (1976).
[Ton88] Tondeur, Ph.: Foliations on Riemannian Manifolds, Springer–Verlag, Berlin (1988).
[Ton97] Tondeur, Ph.: Geometry of Foliations, Birkhäuser, Basel (1997).
[TV90] Tondeur, Ph. and Vanhecke, L.: Transversally symmetric Riemannian foliations, Tôhoku Math.J., **42**, 307–317 (1990).
[TV96] Tondeur, Ph. and Vanhecke, L.: Jacobi fields, Riccati equation and Riemannian foliations, Illinois J. Math., **40**, No. 2, 211–225 (1996).
[Vai71] Vaisman, I.: Variétés riemanniennes feuilletées, Czechoslovak Math. J., **21** (96), 46–75 (1971).
[VG26a] Vrănceanu, G.: Sur les espaces non holonomes, C.R. Acad. Sci. Paris, **183**, 852–854 (1926).
[VG26b] Vrănceanu, G.: Sur le calcul différentiel absolu pour les variétés non holonomes, C.R. Acad. Sci. Paris, **183**, 1083–1085 (1926).
[VG31] Vrănceanu, G.: Sur quelques points de la théorie des espaces non holonomes, Bul. Fac. St. Cernăuţi, **5**, 177–205 (1931).
[VG57] Vrănceanu, G.: Leçons de Géométrie Différentielle, Vol. II, Edition de L' Académie de la République Populaire de Roumanie (1957).
[Wa84] Walczak, P.: Mean curvature functions for codimension–one foliations with all leaves compact, Czech . Math. J., **34**, 146–155 (1984).
[Wal50a] Walker, A.G.: Canonical form for a Riemannian space with a parallel field of null planes, Quart. J. Math. Oxford, **1** (2), 69–79 (1950).
[Wal50b] Walker, A.G.: Canonical forms (II): Parallel partially null planes, Quart. J. Math. Oxford, **1** (2), 147–152 (1950).
[Wal55] Walker, A.G.: Connextions for parallel distributions in the large, Quart. J. Math. Oxford, **6** (2), 301–308 (1955).
[Wal58] Walker, A.G.: Connections for parallel distributions in the large, II, Quart. J. Math. Oxford, **9** (2), 221–231 (1958).
[Was97] Walschap, G.: Umbilic foliations and curvature, Illinois J. Math., **41**, No. 1, 122–128 (1997).
[Wei71] Weinstein, A.: Symplectic manifolds and their Lagrangian submanifolds, Advances in Mathematics, **6**, 329–346 (1971).
[Wil56] Willmore, T.J.: Parallel distributions on manifolds, Proc. London Math. Soc., **6** (3), 191–204 (1956).
[Wol67] Wolf, G.: Spaces of Constant Curvature, McGraw Hill, New York (1967).

[Wu64] Wu, H.: On the de Rham decomposition theorem, Illinois J. Math., **8**, 291–311 (1964).

[YK82] Yano, K. and Kon, M.: Contact CR–submanifolds, Kodai Math. J., **5**, 238–252 (1982).

[YK83] Yano, K. and Kon, M.: CR–Submanifolds of Kählerian and Sasakian Manifolds, Birkhäuser, Boston (1983).

[YK84] Yano, K. and Kon, M.: Structures on Manifolds, World Scientific, Singapore (1984).

[YP40] Yano, K. and Petrescu, S.: Sur les espaces métriques non holonomes complémentaires, Disquisit. Math.Phys., **1**, 191–246 (1940).

[Y83] Yorozu, S.: Behaviour of geodesics in foliated manifolds with bundle–like metrics, J. Math. Soc. Japan, Vol. **35**, No. 2, 251–272 (1983).

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